

Abelian Varieties 2/28-Notes taken by John Meth

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Theorem 0.1. *If L is a positive definite line bundle on a torus X , of type $(d_1, \dots, d_g) \in \mathbb{Z}_{>0}^g$, then $\dim H^0(X, L) = d_1 \cdots d_g$.*

We'll set up some notation and formulas, proved in previous lectures, numbered so they can be referenced throughout the lecture. First note that L is positive definite if and only if $c_1(L) = H$, the hermitian form, is positive definite, i.e. $ImH = E$ is positive definite.

Let $H \in NS(X)$ be nondegenerate, $\pi : V \rightarrow X$ the universal cover of X . Decompose the kernel of π , Λ , into isotropic subspaces with respect to $E = ImH$. Then

$$\Lambda = \Lambda_1 \oplus \Lambda_2 \text{ where } \Lambda_1 = \mathbb{Z} \langle \lambda_1, \dots, \lambda_g \rangle \text{ and } \Lambda_2 = \mathbb{Z} \langle \mu_1, \dots, \mu_g \rangle$$

where $\{\lambda_1, \dots, \lambda_g, \mu_1, \dots, \mu_g\}$ forms a symplectic basis. Let $L \in Pic(X)$ be any line bundle such that $c_1(L) = H$, then $\phi_H = \phi_L : X \rightarrow \hat{X}$ is well defined, since it depends only on H , and is an isogeny, since H is nondegenerate. Now

$$k(L) = ker(\phi_H) = \Lambda(L)/\Lambda \text{ where } \Lambda(L) = \{x \in V | E(x, \Lambda) \subseteq \mathbb{Z}\} = \Lambda(L)_1 \oplus \Lambda(L)_2.$$

Note that

$$\Lambda(L)_1 = \left\{ \frac{n_1}{d_1} \lambda_1 + \cdots + \frac{n_g}{d_g} \lambda_g \mid n_i \in \mathbb{Z} \right\}, \quad \Lambda(L)_2 = \left\{ \frac{n_1}{d_1} \mu_1 + \cdots + \frac{n_g}{d_g} \mu_g \mid n_i \in \mathbb{Z} \right\},$$

and thus

$$k(L) = k(L)_1 \oplus k(L)_2 \text{ where } k(L)_i = \Lambda(L)_i / \Lambda_i = \mathbb{Z}/d_1 \oplus \cdots \oplus \mathbb{Z}/d_g.$$

Recall that the above decomposition decomposes V into $V_1 \oplus V_2$, and that H restricted to $V_2 \times V_2$ is a symmetric real form. We'll need

(1)

$$H|_{V_2 \times V_2}(V_2 \times V_2) \subseteq \mathbb{R}$$

(2)

$$E|_{V_2 \times V_2} = 0, \quad E|_{V_1 \times V_1} = 0, \text{ and}$$

(3)

$$E(\lambda_i, \mu_j) = d_i \delta_{ij}.$$

By extending scalars to \mathbb{C} we get $B : V \times V \rightarrow \mathbb{C}$ which is C -bilinear and symmetric. Notice that under the extension of scalars we have a different decomposition of V into $V_2 \oplus iV_2$. For $v, w \in V$, we write

$$v = v_2 + iv'_2 \text{ and } w = w_2 + iw'_2 \text{ with } v_2, v'_2, w_2, w'_2 \in V_2 = \Lambda_2 \otimes \mathbb{C}$$

and obtain formulas

$$(4) \quad (H - B)(v, w) = 2H(v'_2, w'_2) - 2iH(v_2, w'_2),$$

$$(5) \quad (H - B)|_{V \times V_2} = 0, \text{ and}$$

$$(6) \quad (H - B)|_{V_2 \times V} = 2iE.$$

Now we identify $H^0(X, L)$ with the space of classical theta functions $\{\Theta(z) | \Theta(z + \lambda) = e_\lambda(z)\Theta(z)\}$ where

$$e_\lambda(z) = \chi_0(\lambda)e^{\pi(H-B)(z, \lambda) + \frac{\pi}{2}(H-B)(\lambda, \lambda)}$$

and

$$\chi_0(\lambda) = e^{\pi i E(\lambda_1, \lambda_2)} \text{ where } \lambda = \lambda_1 + \lambda_2 \in \Lambda_1 \oplus \Lambda_2.$$

Notice that we can assume here that $L = L_0 = L(H, \chi_0)$ since any other L with $c_1(L) = H$ is a translation of L_0 . For this factor of automorphy

$$e_{\lambda_2}(z) = 1 \text{ for } \lambda_2 \in \Lambda_2 \Rightarrow \Theta(z + \lambda_2) = \Theta(z),$$

i.e. Θ is periodic with respect to Λ_2 .

Claim 0.2. Θ has a Fourier expansion

$$\Theta(z) = \sum_{\lambda \in \Lambda(L)_1} c_\lambda e^{\pi(H-B)(z, \lambda)}, \quad c_\lambda \in \mathbb{C}.$$

Proof. Since $\Theta(z)$ is periodic with respect to Λ_2 , we know

$$\Theta(z) = \sum_{\vec{n}=(n_1, \dots, n_g) \in \mathbb{Z}^g} c_{\vec{n}} e^{2\pi i(n_1 z_1 + \dots + n_g z_g)}$$

by Fourier. Let $z = x + iy$ and $\lambda = a + ib$, $x, y, a, b \in V_2$. Then

$$\begin{aligned} (H - B)(z, \lambda) &= (H - B)(x, \lambda) + (H - B)(iy, \lambda) \\ &= 2iE(x, \lambda) + (H - B)(iy, \lambda) \text{ (by (6))} \\ &= 2iE(x, \lambda) + 2H(y, b) \text{ (by (4))} \\ &= 2iE(x, \lambda) + 2E(iy, b) \text{ (by (1))} \\ &= 2iE(x, \lambda) - 2E(y, ib) \\ &= 2iE(x, \lambda) - 2E(y, \lambda) \text{ (by (2)).} \end{aligned}$$

Now express x and y in the basis of V_2 , $x = \sum_i x_i \mu_i$, $y = \sum_i y_i \mu_i$, and let $\lambda = \frac{n_1}{d_1} \lambda_1 + \cdots + \frac{n_g}{d_g} \lambda_g \in \Lambda(L)_1$. Then

$$\begin{aligned} (H - B)(z, \lambda) &= 2iE(x, \lambda) - 2E(y, \lambda) \\ &= -2i(n_1 x_1 + \cdots + n_g x_g) + 2(n_1 y_1 + \cdots + n_g y_g) \text{ (by (3))} \\ &= -2i \sum_{i=1}^g n_j z_j. \end{aligned}$$

Thus

$$\sum_{\lambda \in \Lambda(L)_1} c_\lambda e^{\pi(H-B)(z, \lambda)} = \sum_{\lambda \in \Lambda(L)_1} c_\lambda e^{2\pi i(n_1 z_1 + \cdots + n_g z_g)} = \Theta(z).$$

□

Now fix $\lambda_1 \in \Lambda_1$, then we have

$$\begin{aligned} \Theta(z + \lambda_1) &= e_{\lambda_1}(z) \Theta(z) \\ \sum_{\lambda \in \Lambda(L)_1} c_\lambda e^{\pi(H-B)(z+\lambda_1, \lambda)} &= \sum_{\lambda \in \Lambda(L)_1} c_\lambda e^{\pi(H-B)(z, \lambda+\lambda_1)} \\ \sum_{\lambda \in \Lambda(L)_1} c_\lambda e^{\pi(H-B)(z+\lambda_1, \lambda)} &= \sum_{\lambda \in \Lambda(L)_1} c_{\lambda-\lambda_1} e^{\pi(H-B)(z, \lambda)} \end{aligned}$$

$\Rightarrow c_{\lambda-\lambda_1} = c_\lambda e^{\pi(H-B)(\lambda_1, \lambda)}$. Thus the coefficients are constant on cosets of $k(L)$ and there are only $|k(L)|$ coefficients. Thus $\dim H^0(X, L) \leq d_1 \cdots d_g$. But we still don't know that it even has positive dimension. Now our hero enters the picture, wielding the Riemann Theta function:

$$\Theta(z) = \sum_{\lambda \in \Lambda_1} e^{\pi(H-B)(z, \lambda) - \frac{\pi}{2}(H-B)(\lambda, \lambda)}.$$

Claim 0.3. $\Theta \in H^0(X, L)$.

Proof. We must check that Θ is entire and satisfies the appropriate condition under translation by the lattice.

First, Θ is entire.

Let

$$f(z) = \sum_{\lambda \in \Lambda_1} |e^{\pi(H-B)(z, \lambda) - \frac{\pi}{2}(H-B)(\lambda, \lambda)}|.$$

We want to show f is bounded on compact sets $K \subseteq V$. We can put K in an open ball $\{z \in V \mid |z| < R\}$. Let $\phi(z, \lambda) = \pi \operatorname{Re}(H - B)(z, \lambda) = 2\pi H(y, b)$. Now H is positive definite if and only if there exists a $c > 0$ such that $\phi(\lambda, \lambda) \geq c|\lambda|^2$ for all $\lambda \in \Lambda$. Now

$$f(z) \leq \sum_{\lambda \in \Lambda_1} e^{\phi(z, \lambda) - 1/2\phi(\lambda, \lambda)} \leq \sum_{\lambda \in \Lambda_1} e^{k|\lambda|R - 1/2c|\lambda|^2} < \infty$$

so Θ is entire.

Second,

$$\Theta(z + \lambda_2) = \Theta(z) \text{ for } \lambda_2 \in \Lambda_2$$

and

$$\Theta(z + \lambda_1) = e_{\lambda_1}(z)\Theta(z) \text{ for } \lambda_1 \in \Lambda_1.$$

The first equality is easy to see using (5). For the second, compute

$$\begin{aligned} e_{\lambda_1}(z)\Theta(z) &= \sum_{\lambda \in \Lambda_1} e^{\pi(H-B)(z,\lambda) - \frac{\pi}{2}(H-B)(\lambda,\lambda) + \pi(H-B)(z,\lambda_1)} \\ &= \sum_{\lambda \in \Lambda_1} e^{\pi(H-B)(z,\lambda+\lambda_1) - \frac{\pi}{2}(H-B)(\lambda,\lambda)} \\ &= \sum_{\lambda \in \Lambda_1} e^{\pi(H-B)(z,\lambda+\lambda_1) - \frac{\pi}{2}(H-B)(\lambda+\lambda_1,\lambda+\lambda_1)} \\ &= \Theta(z + \lambda_1). \end{aligned}$$

□

Now we want to create more theta functions. For $u \in \Lambda(L)$ let

$$\Theta_u(z) = \frac{1}{e_u(z)}\Theta(z + u) \text{ where } e_u(z) = \chi_0(u)e^{\pi(H-B)(z,u) + \frac{\pi}{2}(H-B)(u,u)} \in H^0(X, L).$$

Note that the cocycle condition extends to $\Lambda(L) \supseteq \Lambda$. Note also that $\Theta_{u+\lambda} = \Theta_u$ for all $\lambda \in \Lambda$, so we can write $\Theta_{\bar{u}} = \Theta_u$, $\bar{u} \in k(L)$.

Claim 0.4. $\{\Theta_{\bar{u}}(z)\}_{\bar{u} \in k(L)}$ forms a basis of $H^0(X, L)$.

Proof. Any $f \in H^0(X, L)$ has a Fourier expansion

$$f(z) = \sum_{\lambda \in \Lambda_1} c_\lambda e^{\pi(H-B)(z,\lambda)}$$

thus the $\Theta_{\bar{u}}$ are independent. □

Remark 0.5. If $L_c = t_c^*L_0$ ($c = \text{character of } L$), let Θ_0^0 be the Riemann theta, and $\Theta_0^c = e^{-\pi H(z,c) - \frac{\pi}{2}H(c,c)}\Theta_0^0(z + c)$. Then $\{\Theta_{\bar{u}}^c = \frac{1}{e_u(z)}\Theta_0^c(z + u)\}_{\bar{u} \in k(L)}$ is a basis for $H^0(X, L_c)$.

Definition 1. A polarized abelian variety (X, H) consists of a torus X and an element $H \in NS(X)$ which is positive definite. The type of this polarization is the type of $\text{Im } H$, i.e. (d_1, \dots, d_g) . If $d_1 = \dots = d_g = 1$, this is called a principally polarized abelian variety (ppav).