1 3/2 - Craig Michoski

1.1 Abelian Varieties and Complex Tori

Definition 1. *A polarized abelian variety is a pair* (X, H)*, with the torus* $X = V/\Lambda$ *and Hermitian form* $H \in NS(X)$ *positive definite of type* $(d_i \dots d_g)$*.*

Definition 2. *An abelian variety is a torus which is polarizable. Polarizations of type* $(1, 1, , \dots, 1)$ *are called principal. Principally polarized abelian varieties are denoted ppav's, with dimension* g*.*

Definition 3. *A morphism of polarized abelian varieties* $F\colon (X, H) \to (Y, M)$ *is a morphism of tori such that* $F^*(M) = H$*.*

Remark 1.1. *1) A subtorus of an abelian variety is an abelian variety*

2) A torus isogenous to an abelian variety is an abelian variety

3) The dual of an abelian variety is an abelian variety.

Proposition 1.2. *Every* 1*-dimensional torus is an abelian variety.*

Proof. Write the torus as $X = \mathbb{C}/\Lambda_\tau$ where $\Lambda_\tau = \mathbb{Z} \times \mathbb{Z}\tau$ with $\Im\tau > 0$. Then $H(z, w) = \frac{z\cdot\bar{\omega}}{\Im\tau}$ is a Hermitian form on $\mathbb{C}$ which satisfies the integrality property. ☐

Remark 1.3. *The general* g*-dimensional torus* $(g \geq 2)$ *is not polarizable. This asks: for* $X = V/\Lambda$ *is there an* $E\colon V \times V \to \mathbb{R}$ *where* $E(ix, iy) = E(x, y)$, $(\Lambda, \Lambda) \subseteq \mathbb{Z}$*? Fix* $\Lambda = \Lambda_1 \oplus \Lambda_2$*, where* $\Lambda_2 = \mathbb{Z}^g : \mu_i = e_i = (0 \dots 1 \dots 0)$, $\lambda_i = (\lambda_{i1} \dots \lambda_{ig})$, $Z = (\lambda_1 \dots \lambda_g) \in M_{g,2g}(\mathbb{C})$*. This gives us* g^2 *dimensions of tori. We want a Hermitian form such that* $\Lambda = \Lambda_1 \oplus \Lambda_2$ *is a decomposition into isotropic subspaces, and we know that every torus is given by* $Z = {}^tZ$ *with* $\Im Z > 0$*, so we have* $\binom{g+1}{2>g^2}$ *moduli for the abelian variety, thus the general torus is not an abelian surface.*

Proposition 1.4. *Every polarized abelian variety is isogenous to a ppav.*

Proof. Take (X, H) a pav of type $(d_1 \dots d_g)$. Fix an isotropic decomposition $\Lambda = \Lambda_1 \oplus \Lambda_2$, $V = V_1 \oplus V_2$, $\Lambda_1 = \mathbb{Z}\langle\lambda_1 \dots \lambda_g\rangle$, $\lambda_2 = \mathbb{Z}\langle\mu_1 \dots \mu_g\rangle$. Take the characteristic zero bundle $L = L(H, \chi_0)$, and create the map $X \to X_1 = V/\Lambda(L)_1 \oplus \Lambda_2$, where $\Lambda(L)_1 = \mathbb{Z}\langle\frac{\lambda_1}{d_1} \dots \frac{\lambda_g}{d_g}\rangle$. Define a polarization on X_1, by $M := L(h, \chi_0|_{\Lambda(L)_1 \oplus \Lambda_2})$, $\chi_0\colon \Lambda \to S^1$, $\chi_0(\lambda) = e^{\pi i E(\lambda_1, \lambda_2)}$ where we extend χ_0 to $\Lambda(L)_1 \oplus \Lambda_2$. The symplectic basis is written $(\frac{1}{d_1}\lambda_1 \dots \frac{1}{d_g}\lambda_g, \mu_1 \dots \mu_g)$, which has the desired type. (see Apel-Humbert theorem) ☐

1.2 Divisors and Linear Series

Take X a complex manifold and consider a (Cartier) divisors on X: $D = \{(U_\alpha, f_\alpha \in M(U_\alpha))\}_\alpha$, such that $\{U_\alpha\}_\alpha$ is an open covering of X with the quotient $f_\alpha/f_\beta \in \mathcal{O}_X(U_{\alpha\beta})$.

Remark 1.5. *0)* $D = \{(U_\alpha, f_\alpha)\}$, $E = \{(U_\alpha, h_\alpha)\}$*, then define* $D + E = \{(U_\alpha, f_\alpha h_\alpha)\}$*.*

1) Globally defined meromorphic functions $f \in M(X)$ *give what are called principal divisors* $D = \{(U_\alpha, f)\}_\alpha = \mathrm{div}(f)$*.*

2) A divisor is called effective, written $D \geq 0$*, if* $D = \{(U_\alpha, f_\alpha)\}$*,* $f_\alpha \in \mathcal{O}_X(U_\alpha)$*.*

3) If $D, E \in \mathrm{Div}(X)$*, we say they are linearly equivalent,* $D \equiv E$*, if and only if* $D = E + \mathrm{div}(f)$ *for some* $f \in M(X)$*. This is an equivalence relation.*

4) To a divisor $D \in \mathrm{Div}(X)$ *we can associate the line bundle of holomorphic functions* $\mathcal{O}_X(D) \in \mathrm{Pic}(X)$*, where* $\{(U_\alpha, f_\alpha)\} \mapsto \mathcal{O}_x(D) = \{(U_{\alpha\beta}, \frac{f_\alpha}{f_\beta})\}$*. This is a group homomorphism, and the principle divisors are in the kernel. When we quotient out by the principal divisors it becomes an isomorphism:* $\mathrm{Div}(X)/\mathrm{PDiv}(X) \xrightarrow{\sim} \mathrm{Pic}(X)$*.*

5) Every divisor is a difference of two effective divisors. Note that

$$\Gamma(X, \mathcal{O}_X(D)) = H^0(X, \mathcal{O}_X(D)) = \{s_\alpha \in \mathcal{O}_X(U_\alpha) : s_\alpha = \frac{f_\alpha}{f_\beta} s_\beta \in U_{\alpha\beta}\}.$$

So $\frac{s_\alpha}{f_\alpha} = \frac{s_\beta}{f_\beta}$ *and we can use these to define* $f \in M(X)$*. Then our space of sections becomes* $\{f \in M(X) : ff_\alpha \in \mathcal{O}_X(U_\alpha)\} = \{f \in M(X) : \mathrm{div}(f) + D \geq 0\}$*.*

Associated to every divisor, D, is a linear series $|D| = \{E \in \mathrm{Div}(X) \mid E \equiv D, E \geq 0\}$. Note that this really only depends on the linear equivalence class of D, so we write $|D| = |\mathcal{O}_X(D)|$. Also note that $|D| = \mathbb{P}(H^0(X, \mathcal{O}_X(D)))$, so it is always a projective space.

The Geometric Picture: Recall that a Weil divisor, W, is a formal combination of hypersurfaces on X. We have $W = n_1 H_1 + \ldots + n_s H_s$ with $n_i \in \mathbb{Z}$ and $H_i \subseteq X$ hypersurfaces locally given by a single equation. Define $\mathrm{Div}(X) \to W\mathrm{Div}(X)$ by $W = \{(U_\alpha, f_\alpha)\} \mapsto D = \sum_{H \subseteq X} \mathrm{ord}_H(f_\alpha) H$; effective divisors have n_i's nonnegative.

Note 1.6. *Let* $H \subseteq X$ *be a hypersurface. Then* $\mathcal{O}_{X,H}$*, the ring of meromorphic functions on* X *that are regular somewhere along* H*, is a DVR with valuation* ord_H*. For example, when* $\mathrm{dim} X = 1$*,* $H = p \in X$*,* $\mathcal{O}_{X,p} = \mathbb{C}\{t\}$ *with the usual order,* ord_p*, from complex analysis.*

Let $L = \mathcal{O}_X(D)$, and suppose $\mathrm{dim} H^0(X, \mathcal{O}_X(D) < \infty$, which will always be true. Pick a basis, $\{s_0 \ldots s_r\}$ for the space of sections $H^0(X, \mathcal{O}_X(D))$. Construct $\phi \colon X \dashrightarrow \mathbb{P}^r$ where $\phi(p) := [s_0(p) \ldots s_r(p)]$ and each $s_i = \{s_{i,\alpha} \in \mathcal{O}_X(U_\alpha) : s_{i,\alpha} = \frac{f_\alpha}{f_\beta} s_{i,\beta} \in U_{\alpha\beta}\}$. ϕ is well-defined, choose α such that $p \in U_\alpha$ and $\phi(p) = [s_{0,\alpha}(p) : \ldots : s_{r,\alpha}(p)]$.

Definition 4. *A linear series* L *is base point free if for all* $p \in X$ *there exists at least one section,* $s \in H^0(X, L)$ *with* $s(p) \neq 0$*.*

We then have a correspondence between base-point free linear series on X with a choice of basis for $H^0(X, L)$ and maps to projective space. The previous remarks who how to get the map to projective space, given the line bundle and basis. Conversely, given $f : X \to \mathbb{P}^r$, there is a line bundle on $\mathbb{P}^r$ that pulls back to one on X that gives the map and the sections, namely $\mathcal{O}_{\mathbb{P}^r}(1)$, which we now describe. Let $\mathcal{O}_{\mathbb{P}^r}(-1) = \{(l, z) \in \mathbb{P}^r \times \mathbb{C}^{r+1} : z \in l\}$, which is a line bundle on $\mathbb{P}^r$. Write $U_i = \mathbb{C}^r = \{[x_0 \ldots x_r] \in \mathbb{P}^r : x_i \neq 0\}$, so that $\{U_i\}$ is an open cover of $\mathbb{P}^r$. Then $\mathcal{O}_{\mathbb{P}^r}(-1) = \{(U_i, \frac{x_j}{x_i})\}$. Define $\mathcal{O}_{\mathbb{P}^r}(1)$ as the dual line bundle, $\{(U_i, \frac{x_i}{x_j})\}$. Note that $H^0(\mathcal{O}_{\mathbb{P}^r}(1)) = \{s_i \in \mathcal{O}_{\mathbb{P}^1}(U_i) : s_i = \frac{x_i}{x_j} s_j\} = \mathbb{C}[x_o \ldots x_r]$. Then $L := f^* \mathcal{O}_{\mathbb{P}^r}(1) = \{f^{-1}(U_i), \frac{f^*(x_i)}{f^*(x_j)}\}$ is a line bundle on X, $s_i = f^*(x_i) \in H^0(X, L)$ is a basis, and $f = [s_0 : \ldots : s_r]$.