# Mirror symmetry for elliptic curves

Matthias Ihl[1] and Alexander Kahle[2]

[1]*Dept. of Physics, University of Texas, Austin, Texas 78712, USA*
[2]*Dept. of Mathematics, University of Texas, Austin, Texas 78712, USA*

This is a termpaper for the M390C Abelian Varieties class taught by Gavril Farkas in Spring 2006 at the University of Texas. We try to provide a survey of basic ideas relating to mirror symmetry in the special case of elliptic curves. This turns out to be the only framework in which most computations can be done explicitly. We will try to highlight both the mathematical and the physical side of the story.

## THE PHYSICS BEHIND MIRROR SYMMETRY

### History

The discovery of mirror symmetry by B. Greene and M.R. Plesser [1] in 1990 has initiated a new era of mutual interaction between physics and mathematics. Lots of progress has been made in terms of better understanding string theory, its compactifications and worldsheet theories by studying (conjectured) mirror pairs of Calabi-Yau manifolds. We will review the relevant physics below and try to make contact with the mathematics. From the mathematics point of view, the rich mathematical structures related to or discovered through mirror symmetry have had a great impact on fields like algebraic geometry.

### $N = 2$ SCFTs

Let us try to outline what mirror symmetry is from a physicist's point of view. Mirror symmetry, generally speaking, relates certain two-dimensional (worldsheet) $N = 2$ superconformal field theories (SCFT). We will not describe in very much detail what such a field theory is; the interested reader may consult [2, 7] and references therein. The following descripyion should suffice for our purposes. An $N = 2$ SCFT is a two-dimensional conformal field theory with an action of (two copies) of the $N = 2$ super-Virasoro algebra on the infinite dimensional vector space of states $V$. These copies are often referred to as left- and right-moving. The generators schematically look like

$$[L, L] \sim L,\ [L, J] \sim J,\ [J, J] \sim c,\ [L, Q^{\pm}] \sim Q^{\pm},$$
$$[J, Q^{\pm}] \sim \pm Q^{\pm},\ [Q^{\pm}, Q^{\pm}] = 0,\ [Q^{\pm}, Q^{\mp}] \sim J + L + c.$$

where $c$ collectively stands for central elements.
An isomorphism between superconformal theories is a one-to-one map that preserves the relevant structures. It is important to note that in most cases, two $N = 2$ superconformal field theories are only isomorphic as $N = 1$ SCFTs, for the following reason: An $N = 2$ super-Virasoro algebra has a so-called mirror automorphism:

$$M : L_n \mapsto L_n,\ J_n \mapsto -J_n, Q_n^{\pm} \mapsto Q_n^{\mp}. \qquad (.1)$$

Suppose we have an isomorphism $g : V \xrightarrow{\sim} W$ that acts as the identity on the right $N = 2$ super-Virasoro algebra but acts as the mirror automorphism on the left $N = 2$ super-Virasoro algebra. This may be called a left mirror morphism of $N = 2$ SCFTs. Physicists would call two $N = 2$ SCFTs "the same" or "mirror symmetric to one another" if there exists a left or right mirror morphism between them. However, two $N = 2$ SCFTs are only isomorphic as $N = 2$ SCFTs if they are both left- and right-mirror symmetric.

### Calabi-Yau manifolds and $N = 2$ SCFTs

As a next step, let us look at the relation between Calabi-Yau (CY) manifolds and $N = 2$ SCFTs. To the geometrical data of a CY manifold (i.e., the complex and (complexified) Kähler structures) one can associate an $N = 2$ non-linear sigma model. This is a classical field theory whose Lagrangian can be explicitly constructed from the data above and whose symmetry group includes (two copies of) the un-extended version of an $N = 2$ super-Virasoro algebra. Quantization of this classical non-linear sigma model should yield an $N = 2$ SCFT; however, the precise procedure is not known in general and may not exist. It is also interesting to study the opposite direction: Namely, to try to reconstruct a CY manifold from a given $N = 2$ SCFT. This reconstruction is, not surprisingly, not unique. However, some features of the CY can be gleaned from the central charge $c$, namely the dimension of the CY, and from the BRST cohomology of the nilpotent operator $D_B := Q_0^+ + \overline{Q}_0^+$, namely the Hodge numbers $h^{(p,q)}$. If two CYs are complex conjugate, then the corresponding $N = 2$ SCFTs are isomorphic by mirror symmetry as $N = 2$ theories (see above). The converse statement is not true in general. We shall adopt the expression "mirror symmetric" to refer to a pair of CY manifolds whose corresponding $N = 2$ theories are mirror. These questions are the subject of intense studies in the physics literature concerning mirror symmetry. The name "mirror symmetry" originally came from the observation that for a mirror pair of CY3-folds $V$ and $W$, we have

$$h^{(p,q)}(V) = h^{(3-p,q)}(W), \qquad (.2)$$

which amounts to a reflection of the Hodge diamond about a $\pi/4$ line. As mentioned earlier, a CY manifold has a moduli space that is comprised of complex structure and symplectic, i.e., (complexified) Kähler structure. It turns out to be a very effective tool to study simpler cases where one of these two structures is kept fixed.

### Weak mirror symmetry

This can be achieved by topological twisting, a notion first introduced by Witten [5], based on earlier ideas [1]. It turns out that there are two interesting subclasses of $N = 2$ SCFTs (of CY manifolds) that are topological in a sense that they do not depend on the worldsheet metric: The so-called A- and B-models.
The crucial idea is to consider the BRST cohomology of $V$ instead of $V$ itself. There are two choices of BRST operators, up to complex conjugation, namely $D_A := Q_0^- + \overline{Q}_0^+$ and $D_B := Q_0^+ + \overline{Q}_0^+$, yielding two different topological field theories for each CY manifold $Y$. The most important feature of these topological theories is that the A-model is invariant under variations of the complex structure moduli, while the B-model is independent of the symplectic structure moduli. Consequently, the state space of the A-model is isomorphic to the de Rham cohomology $H^*_{dR}(Y)$, while the B-model state space is isomorphic to the Dolbeault cohomology $H^*(\Lambda^* T^{1,0} Y)$.
Let us now dicuss briefly the notion of weak mirror symmetry: It can be shown that mirror symmetry acts in a very simple fashion on the A- and B-models. Suppose $Y$ and $\widetilde{Y}$ are a mirror pair of CY manifolds in the sense defined above. Then the A-model on $Y$ will be isomorphic to the B-model on $\widetilde{Y}$, since the mirror automorphism simply exchanges $D_A$ and $D_B$. Weak mirror symmetry is a very beautiful and useful notion, since we do not need to worry about quantization of $N = 2$ sigma models. However, it is obvious that a lot of information is lost by making the transition to a topological sector, i.e., by topologically twisting the theory.

### Homological mirror symmetry

This is where the homological mirror conjecture (HMC) of M. Kontsevich [8] enters the game. In 1994, he proposed a extended version of the A- and B-models: Kontsevich conjectured that a suitable extension of the B-model is the bounded derived category of coherent sheaves on $Y$, denoted $\mathbf{D}^b(Y)$, and that the corresponding object associated to the A-model should be some version of the Fukaya category of $Y$, denoted by $\mathbf{D}\mathcal{F}_0(Y)$. More precisely, the homological mirror conjecture states that if $Y$ and $\widetilde{Y}$ are mirror, then $\mathbf{D}^b(Y)$ is the same as $\mathbf{D}\mathcal{F}_0(\widetilde{Y})$ and vice versa.
The precise mathematical definition of these two categories will be discussed in the mathematics section of the paper below.
Here, I want to discuss the physical meaning of the HMC. A good review of this topic can be found in [2, 7, 9].

### $D$-branes

The main idea is to extend the notion of admissible two dimensional worldsheet topological field theories, namely to allow for boundaries of the worldsheet. This leads to the inclusion of $D$-branes, which happen to be fully consistent objects from a $N = 2$ SCFT point of view. They haved played a crucial role in the development of string theory in the past 11 years.
Classically speaking, a $D$-brane is simply a choice of Dirichlet boundary condition on some fields living on the worldsheet (open strings). Now, there are again two types of $D$-branes (boundary conditions) that may be associated with the A-model and the B-model, respectively. We would like for the boundary condition to preserve as many symmetries of the classical field theory as possible. It is, however, not possible to preserve both (left and right) copies of the $N = 2$ super-Virasoro algebra, but many $D$-branes actually preserve a subalgebra. $D$-branes that preserve the diagonal subalgebra are called $D$-branes of type B, or B-branes for short; they turn out to be related to the B-model (hence the name). We can employ the mirror automorphism $M$ above to arrive at boundary conditions that preserve a different subalgebra of the $N = 2$ super-Virasoro algebra. Those $D$-branes are called $D$-branes of type A, or A-branes, and are related to the A-model as will be explained momentarily. The subalgebra in question is given by the following linear combination of generators:

$$Q_n^- + \overline{Q}_n^+,\ Q_n^+ + \overline{Q}_n^-,\ L_n + \overline{L_n},\ -J_n + \overline{J_n}, \quad \forall n \in \mathbb{Z}. \qquad (.3)$$

As mentioned above in the discussion of weak mirror symmetry, the A- and B-models are essentially obtained by passing to the BRST cohomology of $D_A$ and $D_B$, respectively. These relevant combinations of supercharges belong to those $N = 2$ super-Virasoro subalgebras that are preserved by the A-branes and B-branes, respectively. Therefore an A-brane gives rise to consistent boundary conditions for the A-twisted model and, in the same fashion, B-branes are consistent in the B-twisted model. Moreover, this picture implies that these A-branes and

B-branes can be given the structure of categories as outlined by Kontsevich. It is however very challenging to attempt a proof of the HMC since a precise, mathematical definition of a $D$-brane is still lacking and only known for some special cases.
As a consequence, the homological mirror conjecture is mathematically ill-defined in general for the reasons stated above: Our ignorance about how to properly quantize the $N = 2$ non-linear sigma models for arbitrary Calabi-Yau manifolds. The only known exception to this, for which the quantized $N = 2$ SCFT is known explicitly, is the case of flat complex tori. In this case, the homological mirror cojecture is well-defined pending a correct definition of the Fukaya category.
Note that we have not mentioned the important issue of equality of certain A-model and B-model correlation functions, which can be found in the references given below.

### Mirror symmetry for complex tori

To describe the physics of $N = 2$ SCFTs related to flat complex tori in detail would be beyond the scope of this note. Instead we will try to convey some of the distinctive features of these theories. First of all, the space of states is given by

$$V = \mathcal{H}_b \otimes \mathcal{H}_f \otimes \mathbb{C}[\Gamma \oplus \Gamma^*], \qquad (.4)$$

where $\mathcal{H}_{b,f}$ are the bosonic and fermionic Fock spaces, respectively and $\mathbb{C}[\Gamma \oplus \Gamma^*]$ is the space of the group algebra over $\mathbb{C}$. $\Gamma \subset W$ is a lattice in the real vector space $W$, while $\Gamma^* \subset W^*$ is the dual lattice. A good choice of representation for the bosonic and fermionic Fock spaces in terms of creation and annihilation operators would be the Fock-Bargmann representation which can be found in the physics literature. For a given choice of complex structure $j$ on $W$ and a corresponding Kähler metric $g$ together with a 2-form $b$, one can explicitly define the left-moving part of the $N = 2$ super-Virasoro algebra $L, J, Q^{\pm}$ and similarly the right-movers $\overline{L}, \overline{J}, \overline{Q}^{\pm}$ in terms of the above-mentioned creation and annihilation operators. Now we may define a mirror morphism $m$ that acts as follows on the generators of the left- and right-moving $N = 2$ super-Virasoro algebras:

$$m(L) = L', m(Q^{\pm}) = Q'^{\mp}, m(J) = -J',$$
$$m(\overline{L}) = \overline{L}', m(\overline{Q}^{\pm}) = \overline{Q}'^{\pm}, m(\overline{J}) = J'.$$

Given a choice of $(j, g, b)$, we construct two different complex structures on $W/\Gamma \times W^*/\Gamma^*$:

$$\mathcal{J}_1(j, b) = \begin{pmatrix} j & 0 \\ bj + j^t b & -j^t \end{pmatrix},$$
$$\mathcal{J}_2(g, j, b) = \begin{pmatrix} -jg^{-1}b & jg^{-1} \\ gj - bjg^{-1}b & bjg^{-1} \end{pmatrix}.$$

Moreover, we observe that there is a natural $\mathbb{Z}$-valued symmetric bilinear form

$$s((w_1, m_1), (w_2, m_2)) = p(w_1, m_2) + p(w_2, m_1),$$
$$w_{1,2} \in \Gamma, m_{1,2} \in \Gamma^*,$$

which is induced by the natural pairing $p : \Gamma \oplus \Gamma^* \to \mathbb{Z}$. Then one can prove the following theorem.

**Theorem 1.** Two $N = 2$ SCFTs with data $(\Gamma, j, g, b)$ and $(\Gamma', j', g', b')$ are mirror to one another, if there exists and isomorphism of lattices $\Gamma \oplus \Gamma^*$ and $\Gamma' \oplus \Gamma'^*$ which takes $s$ into $s'$, $\mathcal{J}_1$ into $\mathcal{J}_2'$ and $\mathcal{J}_2$ into $\mathcal{J}_1'$.

It turns out that this theorem is a very powerful tool in constructing mirror pairs of flat complex tori. Actually, there is a procedure that turns out to be a special case of T-duality, which is a well-studied duality of superstring theories, that is powerful enough to always produce a complex torus with a flat (complexified) Kähler metric that is mirror to the original torus (for more details cf. [3, 6, 7]).
This concludes our brief survey of the physics of mirror symmetry with a view towards complex tori.

## THE MATHEMATICS OF MIRROR SYMMETRY

### The basic statement of mirror symmetry

Mirror symmetry arose in string theory as a duality between different models in of the theory. This hinted at a mathematical statement where certain categories should be equivalent in mirror pairs of Calabi-Yau manifolds. A precise statement of this was made by Kontsevich at the 1994 ICM[8]. In essence, this says that the derived category of coherent sheaves on a Calabi-Yau manifold $M$ is equivalent to the Fukaya category on its mirror dual, $\widetilde{M}$. However, this statement remains to be proved in generality.

The situation is rather different when one restricts one's attention to elliptic curves. Not only does Kontsevich's statement become very concrete, but it may be explicitly shown to be true.

The remainder of this section will be aimed at elucidating the statement of mirror symmetry for elliptic curves and making it precise. The following section will do mirror symmetry by example: the correspondence will be computed in certain simple cases. It is hoped that this will give the reader a feel for how mirror symmetry actually works. The final section sketches Polishchuk and Zaslow's proof of categorical mirror symmetry for elliptic curves. The entire article follows [4] rather closely.

*Elliptic curves*

If we are to discuss mirror symmetry for elliptic curves, we must start by discussing elliptic curves. An elliptic curve is simply a quotient of $\mathbb{C}$ by a full lattice. Without loss of generality, the lattice may be assumed to be generated by 1 and a complex number $\tau$ in the upper half plane. It is often more convenient to think of the curve as $\mathbb{C}^*/\mathbb{Z}$, where $\mathbb{Z}$ acts via $u \to qu$, and $q = e^{2\pi i\tau}$. We will use the second description extensively, and we shall write $\mathcal{E}_q$ for the elliptic curve with parameter $q$.Note that the choice of $\tau$ (or $q$) is not unique; indeed, there is an action of $SL(2,\mathbb{Z})$ under which the $\tau$ orbits uniquely specify the curve.

**Vector bundles on elliptic curves.** Let us collect some facts about vector bundles on elliptic curves which we shall need later. Any line bundle on $\mathcal{E}_q$ pulls back to a trivial bundle on $C^*$, and as vector bundles are determined from line bundles by successive extensions, this also holds true for them. We may thus construct any vector bundle on $\mathcal{E}_q$ as follows. We choose a (finite dimensional) vector space $V$, and a holomorphic function $A : \mathbb{C}^* \to GL(V)$. The vector bundle $F_q(V, A) \to \mathcal{E}_q$ is then defined as

$$F_q(V,\, A) = C^* \times V/(u,\, v) \sim (uq,\, A(u)\cdot v).$$

Thus one may classify holomorphic vector bundles on $\mathcal{E}_q$ by classifying holomorphic functions to $GL(V)$ up to the equivalence $A(u) \sim B(qu)A(u)B(u)^{-1}$, with $B : C^* \to GL(V)$.

Let us now restrict our attention to line bundles. By the above, a line bundle is given by a holomorphic function $\phi : \mathbb{C}^* \to \mathbb{C}^*$. In this case, we write $L_q(\phi) = F_q(\mathbb{C}, \phi)$. There is a very important line bundle $L \equiv L_q(\phi_0)$, which has $\phi_0(z) = \exp(-i\pi\tau - 2\pi i z) = q^{-1/2}u^{-1}$. Indeed, every holomorphic line bundle on $E_q$ is of the form $t_x^* L \otimes L^{n-1}$ for some $n \in \mathbb{Z}$ and $x \in \mathcal{E}_q$, where $t_x$ the translation map on $\mathcal{E}_q$. Defining

$$\theta[c',\, c''](\tau,\, z) = \sum_{m\in\mathbb{Z}} \exp\left[2\pi i\tau(m+c')^2/2 + 2\pi i(m+c')(z+c'')\right],$$

we see that $\theta(\tau, z) \equiv \theta[0, 0](\tau, z)$ is the pullback to $\mathbb{C}$ the unique (non-zero) section of $L_q$, with $q = e^{2\pi i\tau}$, and that $\theta[a/n, 0](n\tau, nz)$, with $a \in \mathbb{Z}/n\mathbb{Z}$, form a basis for the global sections of $L^n$.

Consider the $r$-fold cover $\pi_r : \mathcal{E}_{q^r} \to \mathcal{E}_q$ which is given by $\pi_r = \rho_{q^r}^{-1} \circ \rho_q$, where $\rho_q : \mathbb{C} \to \mathcal{E}_q$ is the quotient map. It is easy to see this is well defined, and $\pi_r^{-1}(u) = \left\{u,\, uq,\, uq^2,\, \cdots,\, uq^{r-1}\right\}$. Let us observe how vector bundles behave via pull-back and push-forward by $\pi_r$. Pull-back is easy: $\pi_r^* F_q(V, A) = F_{q^r}(V, A^r)$, as easily verified by staring at the definition of $F_q(V, A)$. Push-forward is not much more difficult: $\pi_{r*}F_{q^r}(V, A) = F_q(V \otimes \mathbb{C}^r, \pi_{r*}A)$, where $\pi_{r*}A(v \otimes e_i) = v \otimes e_{i+1}$ for $i = 1, \cdots, r-1$, and $\pi_{r*}A(v \otimes e_r) = Av \otimes e_1$. It is easy to verify the natural isomorphisms

$$\begin{aligned} \pi_{r*}(F_1 \otimes \pi_r^* F_2) &\cong \pi_{r*}(F_1) \otimes F_2, \\ \pi_{r*}(F)^* &\cong \pi_{r*}(F^*), \\ H^0(E_q,\, \pi_{r*}(F)) &\cong H^0(\mathcal{E}_{q^r}, F). \end{aligned}$$

Thus

$$\begin{aligned} \mathrm{Hom}\,(F_1,\, \pi_{r*}F_2) &\cong \mathrm{Hom}\,(\pi_r^* F_1,\, F_2)\,, \\ \mathrm{Hom}\,(\pi_{r*}F_1,\, F_2) &\cong \mathrm{Hom}\,(F_1,\, \pi_r^* F_2)\,. \end{aligned}$$

**Some technical results.** Let us collect some technical results we shall need in proving mirror symmetry for elliptic curves. These will be stated without proof.

**Proposition .1** *Every indecomposable bundle on $\mathcal{E}_q$ is isomorphic to a bundle of the form $\pi_{r*}(L_{q^r}(\phi) \otimes F_{q^r}(\mathbb{C}^k, \exp N)$, where $N$ is a constant indecomposable nilpotent matrix, $\phi = t_x^*\phi_0 \cdot \phi_0^{n-1}$ for some $n \in \mathbb{Z}$ and $x \in \mathbb{C}^*$.*

**Proposition .2** *Let $\phi = t_x^*\phi_0 \cdot \phi_0^{n-1}$, with $n > 0$. Then for any nilpotent endomorphism $N \in \mathrm{End}\,(V)$, there is a canonical isomorphism*

$$\mathcal{V}_{\phi,\,N} : H^0(L(\phi)) \otimes V \to H^0(L(\phi) \otimes F(V,\, \exp N)),$$

*given by*

$$\mathcal{V}_{\phi,\,N}(f \otimes v) = \exp(DN/n)f \cdot v,$$

*where $D = -u\frac{d}{du} = -\frac{1}{2\pi i}\frac{d}{dz}$.*

**Proposition .3** *Let $\phi_1 = t_{x_1}^*\phi_0 \cdot \phi_0^{n_1-1}$, $\phi_2 = t_{x_2}^*\phi_0 \cdot \phi_0^{n_2-1}$, and $N_i \in \mathrm{End}\,(V_i)$, $i = 1, 2$ be nilpotent endomorphisms. Then*

$$\begin{aligned} &\mathcal{V}_{\phi_1,\,N_1}(f_1 \otimes v_1) \circ \mathcal{V}_{\phi_2,\,N_2}(f_2 \otimes v_2) = \\ &\quad \mathcal{V}_{\phi_1\phi_2,\,N_1+N_2}(f_1 \otimes v_1) \\ &\left[\exp\left(\tfrac{n_2N_1 - n_1N_2}{n_1+n_2}\tfrac{D}{n_1}\right)(f_1)\exp\left(\tfrac{n_1N_2 - n_2N_1}{n_1+n_2}\tfrac{D}{n_2}\right)(f_2)(v_1 \otimes v_2)\right], \end{aligned}$$

*where $N_1$, $N_2$ denote $N_1 \otimes 1$, $1 \otimes N_2$ respectively on the right hand side, and $\circ$ denotes the natural composition of sections*

$$\begin{aligned} &H^0(L(\phi_1) \otimes F(V_1, \exp N_1)) \otimes H^0(L(\phi_2) \otimes F(V_2, \exp N_2)) \\ \to \quad &H^0(L(\phi_1\phi_2) \otimes F(V_1 \otimes V_2,\, \exp(N_1 \otimes 1 + 1 \otimes N_2))). \end{aligned}$$

*Regarding $e^{\frac{d}{dz}}$ as the generator of translations, so that (formally) $e^{N\cdot\frac{d}{dz}}f(z) = f(z+N)$, one may re-write the above formula as*

$$\begin{aligned} &\mathcal{V}(f_1 \otimes v_1) \circ \mathcal{V}(f_2 \otimes v_2) = \\ &\mathcal{V}\Bigg(f_1\left(z + \frac{n_1N_2 - n_2N_1}{2\pi i n_1(n_1+n_2)}\right)f_2\left(z + \frac{n_2N_1 - n_1N_2}{2\pi i n_2(n_1+n_2)}\right)(v_1 \otimes v_2)\Bigg) \\ &= \mathcal{V}\Big(f_1\Big(ue^{\frac{n_1N_2 - n_2N_1}{n_1(n_1+n_2)}}\Big)f_2\Big(ue^{\frac{n_2N_1 - n_1N_2}{n_2(n_1+n_2)}}\Big)(v_1 \otimes v_2)\Big). \end{aligned}$$

*Categories*

The statement of categorical mirror symmetry for elliptic curves is naturally made in the language of categories. It would thus be well to spend some time reviewing the basic concepts of category theory.

**Definition .4 (Category)** *A* category $C$ *consists of*

- *A class* Obj $(C)$ *consisting of the* objects *in the category.*
- *A class* Hom $(C)$ *consisting of the morphisms of the category. Each morphism $f$ has a unique source and target object and is usually written $f : a \to b$.*
- *There is composition of morphisms, viz. there is an operation* $\mathrm{Hom}\,(a,\, b) \times \mathrm{Hom}\,(b,\, c) \to \mathrm{Hom}\,(a,\, c)$ *that takes $f : a \to b$ and $g : b \to c$ to their composition $g \circ f : a \to c$.*

*obeying*

- (associativity) *Given $f : a \to b$, $g : b \to c$, $h : c \to d$, one has $(h \circ g) \circ f = h \circ (g \circ f)$.*
- (identity) *For every object $x$ there exists a morphism $1_x$ such that for every morphism $f : a \to b$, $f \circ 1_a = 1_b \circ f$.*

The notation is suggestive, and indeed most natural examples of categories consist of mathematical objects with "nice" maps between them. For example, the category **Set** has all sets as objects, and maps between sets as morphisms.

The correct notion of a map between categories is furnished by a *functor*.

**Definition .5** *A* functor $F$ *from category $C$ to category $D$ (henceforth written $F : C \to D$) assigns to each element $x \in \mathrm{Obj}\,(C)$ an element $F(x) \in \mathrm{Obj}\,(D)$, and to each morphism $f \in \mathrm{Hom}\,(a,\, b)$ a morphism $F(f) \in \mathrm{Hom}\,(F(a),\, F(b))$ satisfying*

- *$F(1_x) = 1_{F(x)}$, and*
- *$F(f \circ g) = F(f) \circ F(g)$ for all $f : b \to c$, $g : a \to b$.*

We need one more concept before we may speak of equivalence of categories.

**Definition .6** *Given two functors $F, G : C \to D$, a* natural transformation $\eta$ *from $F$ to $G$ assigns to each object $x \in \mathrm{Obj}\,(C)$ a morphism $\eta_x : F(x) \to G(X)$ such that for every $f : x \to y$ one has $\eta_y \circ F(f) = G(f) \circ \eta_x$. If each $\eta_x$ is an isomorphism then $\eta$ is said to be a* natural isomorphism*.*

Having defined functors, and the concept of natural transformation between them, we may now formulate the notion of equivalence between categories:

**Definition .7** *An* equivalence of categories *consists of functors $F : C \to D$ and $G : D \to C$, and natural isomorphisms $\eta : FG \to 1_C$ and $\varepsilon : GF \to 1_D$. One usually only explicitly states part of the data: one says $F : C \to D$ "is" an equivalence of categories if there exists an inverse $G$ and the natural isomorphisms as above exist.*

**$A^\infty$ categories.** $A^\infty$ categories are a weakening of the notion of categories, where one allows the composition of morphisms not to be associative. Formally:

**Definition .8** *An $A^\infty$ category $C$ consists of*

- *A class of objects* Obj $(C)$,
- *For any $x$, $y \in \mathrm{Obj}\,(C)$, a $\mathbb{Z}$-graded abelian group of morphisms* Hom $(x,\, y)$,
- *Composition maps*

$$m_k : \mathrm{Hom}\,(x_1,\, x_2) \otimes \mathrm{Hom}\,(x_1,\, x_2) \otimes \ldots \otimes \mathrm{Hom}\,(x_k,\, x_{k+1}) \to \mathrm{Hom}\,(x_1,\, x_{k+1}),$$

*for $k \geq 1$ of degree $2 - k$ satisfying*

$$\sum_{r=1}^{n} \sum_{s=1}^{n-r+1} (-1)^{\varepsilon} m_{n-r+1} (a_1 \otimes \ldots \otimes a_{s-1} \otimes m_r(a_s \otimes \ldots \otimes a_{s+r-1}) \otimes a_{s+r} \otimes \ldots \otimes a_n) = 0$$

*for every $n \geq 1$, where $\varepsilon = (r+1)s + r(n + \sum_{j=1}^{s-1} \deg(a_j))$.*

The last condition may appear somewhat mysterious, and replaces the associative condition in an ordinary category. Let us try to understand this condition by examining it in particular cases. For $n = 1$, it states

$$m_1 \circ m_1(x) = 0.$$

Notice that this, along with $\deg(m_1) = 1$ shows that $m_1$ is a derivation on the space of homomophisms, with respect to which one may take the cohomology. Let us thus denote $m_1$ by $d$. We shall see that taking $H^0$ of an $A^\infty$-category allows one to recover an (ordinary) cartegory underlying the $A^\infty$-category. To whit, let us examine the condition when $n = 2$. It then states $m_2$ is degree zero and satisfies

$$m_2(d(a_1) \otimes a_2) + (-1)^{\deg(a_1)} m_2(a_1 \otimes d(a_2)) - d(m_2(a_1 \otimes a_2)) = 0.$$

Thus $m_2$ is a chain map, and induces a product on cohomology. Finally, $n = 3$ shows

$$\begin{aligned} &d(m_3(a_1 \otimes a_2 \otimes a_3)) + m_2(m_2(a_1 \otimes a_2) \otimes a_3) \\ &- m_2(a_1 \otimes m_2(a_2 \otimes a_3)) + m_3(d(a_1) \otimes a_2 \otimes a_3) \\ &+ (-1)^{\deg(a_1)} m_3(a_1 \otimes d(a_2) \otimes a_3) \\ &+ (-1)^{\deg(a_1) + \deg(a_2)} m_3(a_1 \otimes a_2 \otimes d(a_3)) = 0, \end{aligned}$$

so that passing to cohomology one obtains

$$m_2(m_2(a_1 \otimes a_2) \otimes a_3) = m_2(a_1 \otimes m_2(a_2 \otimes a_3)),$$

in other words, multiplication (a.k.a. composition) is associative at the level of cohomology.

Thus one may think of $A^\infty$-categories in terms of the following analogy:

$$A^\infty\text{-categories} : \text{categories} :: \\ \text{co-chain complexes} : \text{cohomology}.$$

*The derived category of coherent sheaves*

The first category involved in categorical mirror symmetry is the derived category of coherent sheaves on a space. It would take us too far afield to understand this object in full generality; instead, we shall simply seek to understand it for elliptic curves.

**Coherent sheaves on elliptic curves.** While the category of coherent sheaves on a general space may be quite complicated, when one restricts one's attention to (complex) curves one finds that it admits a particularly simple description. There are two classes of coherent sheaves that spring readily to mind: vector bundles and "skyscraper"-like sheaves. The latter are defined by the exact sequence

$$\mathcal{O} \xrightarrow{(z-z_0)^n} \mathcal{O} \longrightarrow \mathcal{O}_{nz_0} \ ,$$

where $\mathcal{O}$ is the sheaf of holomorphic functions on the curve, and the map $\mathcal{O} \to \mathcal{O}$ is multiplication by $(z-z_0)^n$. Thus $\mathcal{O}_{nz_0}$ consists of equivalence classes of functions, with functions being thought equivalent if they agree up to their $(n-1)$'st derivative. Scyscraper sheaves may also be formed as equivalence classes of sections of vector bundles. Indeed, for every point $z_0$ on the curve, vector bundle $V$ and nilpotent $N \in \mathrm{End}\,(V|_{z_0})$ there exists such a sheaf. Are there any other coherent sheaves on a curve? The short answer is that there are not: every coherent sheaf on a curve is a direct sum of vector bundles and skyscraper-like sheaves (morphisms are the usual morphisms of sheaves).

**$\mathcal{D}^b\,(E)$.** So what, then, is the (bounded) *derived* category of coherent sheaves on an elliptic curve $E$, $\mathcal{D}^b\,(E)$? In general it is obtained from the category of (bounded) complexes of coherent sheaves by inverting quasi-isomorphisms. However, for curves there is a much more down-to-earth description: it is simply a direct sum of objects of the form $F[n]$, where $F[n]$ is a complex with the only non-zero term in degree $-n$, and this non-zero term equal to $F$, a coherent sheaf on the curve. Thus every object in $\mathcal{D}^b\,(E)$ consists of a direct sum of objects of the form $F[n]$, with $F$ either a vector bundle or a skyscraper-like sheaf.

*Fukaya's category*

The second category involved in categorical mirror symmetry is Fukaya's category. As with the derived category of coherent sheaves, it is in general quite difficult to understand, but admits a particularly simple description for elliptic curves. The objects of the Fukaya category is defined on Calabi-Yau manifolds, and for such a manifold $M$, $\mathrm{Obj}\,(\mathcal{F}\,(M))$ consists of pairs $\mathcal{U} = (\mathcal{L},\,\mathcal{E})$, where $\mathcal{L}$ is a special Lagrangian submanifold of $M$, and $\mathcal{E} \to \mathcal{L}$ is a flat bundles (called a local system), along with a lift of the Gauss map $\mathcal{L} \to \mathcal{V}$ (where $\mathcal{V}$ is the bundle of Lagrangian planes over $\mathcal{L}$) to the universal cover of $\mathcal{V}$.

This description may appear forbidding, but for elliptic curves it is quite simple. In this case, minimal Lagrangian submanifolds are just geodesics (with rational slope). Thus they are parameterised by a pair of integers $(p,\,q)$ representing the slope, and a point of intersection with the line $x = 0$ (or $y = 0$ if $p = 0$). In the simplest case the rank of the unitary system is one, and so is a flat line bundle, which is specified by simply giving the monodromy around the curve. The lift of the Gauss map may also be simply stated in this case: the rational slope may be written as a complex phase with rational tangency, ie $e^{i\alpha}$, with $2\pi\alpha \in \mathbb{Q}$ – choosing a lift of the Gauss map amounts to knowing $\alpha$ itself, not just the slope.

The space of morphisms between two objects in the Fukaya category is defined as

$$\mathrm{Hom}\,(\mathcal{U}_i,\,\mathcal{U}_j) = \mathbb{C}^{\#\{\mathcal{L}_i \cap \mathcal{L}_j\}} \otimes \mathrm{Hom}\,(\mathcal{E}_i,\,\mathcal{E}_j)\,,$$

where $\mathrm{Hom}\,(\mathcal{E}_i,\,\mathcal{E}_j)$ is the space of homomorphisms of the vector spaces underlying the local system at the point of intersection. The space of morphisms is $\mathbb{Z}$-graded, by the Maslov index of the intersections.

Let us try to understand this in our setting. For simplicity, we consider $\mathcal{U}_i$, $\mathcal{U}_j$ where the lines go through the origin, and the local systems are flat line bundles. The slopes of the lines are given by $\tan\alpha_i = q/p$, $\tan\alpha_j = s/r$ (with $(p,\,q)$ and $(r,\,s)$ co-prime). Looking at the universal cover of the curve, we see that the lines lift to an infinite family of parallell lines, and inspection shows that there are exactly $|ps - qr|$ non-equivalent points of intersection. We need to specify an element of $\mathrm{Hom}\,(\mathbb{C},\,\mathbb{C})$ at each point of intersection, i.e. a complex number $T_i$ at each point (were the local system rank $n$, the $T_i$'s would be $n \times n$ matrices).

The grading is also easy to understand in this case: it is constant for all points of intersection, as the points are all related by translation. For any point $p \in \mathcal{L}_i \cap \mathcal{L}_j$ the grading is given by

$$\mu(p) = -[\alpha_j - \alpha_i],$$

where $[x]$ represents the greatest integer less than $x$.

Fukaya's category is not in fact a true category, but one in the $A^\infty$ sense. Let us try to understand Fukaya's prescription for the $A^\infty$-structure: An element $u_i$ in $\mathrm{Hom}\,(\mathcal{U}_i, \mathcal{U}_j)$ is represented by a pair $(t_i, a_i)$, where $a_i \in \mathcal{L}_i \cap \mathcal{L}_j$, and $t_i \in \mathrm{Hom}\,(\mathcal{E}_i|_{a_i}, \mathcal{E}_j|_{a_i})$. Bearing this in mind, define

$$m_k(u_1 \otimes \ldots \otimes u_k) = \sum_{a_{k+1} \in \mathcal{L}_1 \cap \mathcal{L}_{k+1}} C(u_1, \ldots, u_k, a_{k+1}) \cdot a_{k+1},$$

where

$$C(u_1, \ldots, u_k, a_{k+1}) = \sum_\phi \pm \exp[2\pi i \int \phi^* \omega] \cdot P \exp[\oint \phi^* \beta],$$

with the sum taken over all (anti-)holomorphic maps $\phi$ (up to projective equivalence) from $D^2$ to $M$ with the following boundary condition: there exist $k+1$ boundary points $p_j = e^{2\pi i \alpha_j}$ such that $\phi(p_j) = a_j$, and $\phi(e^{2\pi i \alpha}) \in \mathcal{L}_j$ for $\alpha \in (\alpha_{j-1}, \alpha_j)$. Here $\omega$ is the complexified Kähler form (it's real part gives the complex structure on $M$ and its imaginary part the hermitian metric), the sign is determined by an orientation in the space of holomorphic maps, $Pe^{\oint}$ represents path-ordered integration around the boundary, and $\beta$ is the connection form of the flat bundle along the boundary. Path-ordered integration is defined by

$$Pe^{\oint \phi^* \beta} = e^{\int_{\alpha_k}^{\alpha_{k+1}} \beta_k d\alpha} \cdot t_k \cdot e^{\int_{\alpha_{k-1}}^{\alpha_k} \beta_{k-1} d\alpha} \cdot t_{k-1} \cdot \ldots \cdot t_1 \cdot e^{\int_{\alpha_{k+1}}^{\alpha_1} \beta_1 d\alpha}.$$

This procedure becomes fairly transparent when we restrict our attention to elliptic curves.

As remarked earlier, to obtain an honest category from the Fukaya category, one needs to take the degree zero homology of the morphisms. With elliptic curves this is easy: one simply takes the degree zero part of $\mathrm{Hom}\,(\mathcal{F}\,(E))$. The resulting category is written $\mathcal{F}^0\,(E)$.

### *The statement of categorical mirror symmetry*

We are almost at the point where categorical mirror symmetry may be precisely stated for elliptic curves. Before we do so we will have to introduce one more concept, the mirror dual of an elliptic curve. Let $E = \mathbb{C}/(\mathbb{Z} \oplus \tau Z)$ be a given elliptic curve with parameter $\tau$. Then its mirror dual curve is $\widetilde{E} = \mathbb{R}^2/(\mathbb{Z} \oplus \mathbb{Z})$ with the Kähler structure to be defined presently.

Recall that a Kähler manifold is a complex manifold with a compatible hermitian metric. This may be conveniently summarised by specifying a (non-degenerate) complex two-form on the space, where the real-part gives the complex structure, and the imaginary part the hermitian metric. In the case of an elliptic curve these must be constant, and one may equivalently specify the Kähler structure with the choice of a complex parameter $\rho = b + iA$, where $b$ is the integral of the complex structure over the torus, and $A$ its area. Then the mirror dual of $E$, $\widetilde{E}$, has Kähler structure given by $\rho = \tau$.

We are finally at the point where we may state the main theorem:

**Theorem .9** *The categories $\mathcal{D}^b\,(E_q)$ and $\mathcal{F}^0\left(\widetilde{E}^q\right)$ are equivalent (where $q = \exp(2\pi i \tau)$).*

Thus mirror symmetry essentially relates vector-bundles (and skyscraper-like sheaves) on one elliptic curve to straight lines on another. The next section will attempt to show exactly what this correspondence is by example. The final section will sketch the general strategy of proof for the statement.

### Mirror symmetry by example

Let us begin with the simplest possible situation: we set $\rho = \tau = iA$, and consider only lines through the origin with trivial local systems (zero holonomy). The basic dictionary will send line bundles of degree $d$ to lines with slope $d$. Consider the lines

$$\begin{aligned} \mathcal{L}_0 &= (1,0), \\ \mathcal{L}_1 &= (1,1), \\ \mathcal{L}_2 &= (1,2). \end{aligned}$$

These will then correspond to line bundles of degree 0, 1, and 2. Thus $\mathcal{L}_0$ corresponds to $\mathcal{O}$, the sheaf of holomorphic functions, and we can choose $\mathcal{L}_1$ to correspond to $L$, so that $\mathcal{L}_2 = L^2$. Recall that the unique global section of $L$ is the theta function $\theta(\tau, z)$. Then (on the derived category side)

$$\begin{aligned} \mathrm{Hom}\,(L_0, L_1) &= H^0(L), \\ \mathrm{Hom}\,(L_1, L_2) &= H^0(L), \\ \mathrm{Hom}\,(L_0, L_2) &= H^0(L^2). \end{aligned}$$

Composition of morphisms (in other words, the product of global sections) gives a map

$$m_2 : H^0(L) \otimes H^0(L) \to H^0(L^2),$$

which may be seen as a factorisation of the product of theta functions into the two global sections of $L^2$.

Let us now try and understand the picture on the Fukaya side. The three lines of slope 0, 1, and 2 all intersect at the origin, which we will denote $e_1$. $\mathcal{L}_0$ intersects $\mathcal{L}_1$ nowhere else, and similarly, $\mathcal{L}_1$ shares only one point with $\mathcal{L}_2$. However, $\mathcal{L}_0$ intersects $\mathcal{L}_2$ at *two* points, the second, denoted by $e_2$, being the image of $(\frac{1}{2}, 0)$. We need to identify these points of intersection with morphisms: for $\mathrm{Hom}\,(\mathcal{L}_0, \mathcal{L}_1)$ we identify $e_1$ with $\theta(\tau, z)$, and similarly for $\mathrm{Hom}\,(\mathcal{L}_1, \mathcal{L}_2)$. For $\mathrm{Hom}\,(\mathcal{L}_0, \mathcal{L}_2)$ we need sections of $L^2$ corresponding to the two points of intersection: choose $e_1 \leftrightarrow \theta[0,\,0](2\tau,\,2z)$, $e_2 \leftrightarrow \theta[\frac{1}{2},\,0](2\tau,\,2z)$. Let us see how this choice checks with Fukaya's prescription for computing $m_2$.

Fukaya tells us that

$$m_2(e_1 \otimes e_1) = C(e_1,\,e_1\,e_1)\cdot e_1 + C(e_1,\,e_1,\,e_2)\cdot e_2,$$

where the $C$'s are computed as certain integrals over discs. Thus, $C(e_1,\,e_1\,e_1)$ is computed by summing all triangles on the universal cover with vertices with the origin as one vertex, the other vertices on lattice points, and edges with slope 0, 1, and 2. The base of the triangle must thus be of integral length, say $n$, and run from $(0,\,0)$ to $(n,\,0)$. The third vertex must be at $(2n,\,2n)$, so that all triangles are indexed by $n$, and have area $n^2A$ (recall that $A$ is the area of the elliptic curve). Thus

$$C(e_1,\,e_1,\,e_1) = \sum_{n=-\infty}^{\infty} \exp[-2\pi A n^2].$$

For $C(e_1,\,e_1,\,e_2)$ one sums over triangles with vertices $(0,\,0)$, $(0,\,n+1/2)$, and $(2n+1,\,2n+1)$. Similar considerations lead one to conclude that

$$C(e_1,\,e_1,\,e_2) = \sum_{n=-\infty}^{\infty} \exp[-2\pi A(n+1/2)^2].$$

Comparing these expressions with the definitions for the classical theta functions then shows

$$\begin{aligned} C(e_1,\,e_1,\,e_1) &= \theta[0,\,0](2iA,\,0) = \theta[0,\,0](2\rho,\,0), \\ C(e_1,\,e_1,\,e_2) &= \theta[1/2,\,0](2iA,\,0) = \theta[1/2,\,0](2\rho,\,0). \end{aligned}$$

Thus Fukaya's prescription for $m_2$ yields

$$\begin{aligned} &\theta(\rho,\,z)\theta(\rho,\,z) = \\ &\theta[0,\,0](2\rho,\,0)\theta[0,\,0](2\rho,\,2z) \\ &\;+\theta[1/2,\,0](2\rho,\,0)\theta[1/2,\,0](2\rho,\,2z), \end{aligned}$$

which is precisely the addition formula.

Let us know examine what happens to the correspondence when $\mathcal{L}_2$ is allowed to move off the origin. If we label the closest (positive) intersection of $\mathcal{L}_2$ with the $x$-axis $\alpha$, we see that $\mathcal{L}_0 \cap \mathcal{L}_2$ still contains two points: $e_{1,\,\alpha} = (\alpha,\,0)$, and $e_{2,\,\alpha} = (1/2+\alpha,\,0)$. Computing the coefficients in $m_2$ for these shifted intersection points, we see

$$\begin{aligned} &C(e_1,\,e_1,\,e_{1,\,\alpha}) = \\ &\sum_{n=-\infty}^{\infty} \exp[-2\pi A(n+\alpha)^2] = \theta[\alpha,\,0](2\rho,\,0), \\ &C(e_1,\,e_1,\,e_{2,\,\alpha}) = \\ &\sum_{n=-\infty}^{\infty} \exp[-2\pi A(n+1/2+\alpha)^2] = \theta[1/2+\alpha,\,0](2\rho,\,0). \end{aligned}$$

We also need to identify where $e_{1,\,\alpha}$ and $e_{2,\,\alpha}$ map to under the correspondence. The obvious guess would be

$$\begin{aligned} e_{1,\,\alpha} &\leftrightarrow \theta[\alpha,\,0](2\tau,\,2z), \\ e_{2,\,\alpha} &\leftrightarrow \theta[1/2+\alpha,\,0](2\tau,\,2z). \end{aligned}$$

This is almost correct: some phases need to be added and then the formula for $m_2$ again becomes equivalent to the addition formula.

Let us now understand what happens when holonomy is added to the local system. We revert to the case where the lines all intersect at the origin, but now allow the bundle over $\mathcal{L}_3$ to have holonomy: that is, we endow it with a connection $d + 2\pi i\beta dt$ (where $t \sim t+1$ is a coordinate on $\mathcal{L}_3$). Each term in the sum defining $C(e_1,\,e_1,\,e_k)$ is now weighted by a factor $exp(2\pi i \oint \beta dt)$, so that

$$\begin{aligned} &C(e_1,\,e_1,\,e_1) = \\ &\sum_{n=-\infty}^{\infty} \exp[-2\pi A n + 2\pi i n\beta] = \theta[0,\,\beta](2\rho,\,0), \\ &C(e_1,\,e_1,\,e_2) = \\ &\sum_{n=-\infty}^{\infty} \exp[-2\pi A(n+1/2)^2 + 2\pi i(n+1/2)\beta] = \theta[1/2,\,\beta](2\rho,\,0). \end{aligned}$$

The corresponding choice of section for $e_1$ and $e_2$ on the right hand side of the formula naturally involves a shift: up to phases $e_1 \leftrightarrow \theta[0,\beta](2\tau, 2z)$ and $e_2 \leftrightarrow \theta[1/2,\beta](2\tau, 2z)$.

So, to summarise, shifting the lines and adding holonomy both induce translations on the Jacobian torus: a shift of the line by $\alpha$ and adding holonomy $2\pi i\beta dt$ causes all the theta functions on the RHS of the addition function to be precisely those that involve the two sections of the degree two line bundle at the point $\alpha\tau + \beta$ on the Jacobian torus. The interesting thing is that on the Jacobian these two translations are on an equal footing, whereas on the mirror dual side they are manifested in rather different ways. This disparity is apparently also evident in the physics motivating mirror symmetry.

Passing to the case where $b \neq 0$ is simple: the same formulae hold true, with the factors $iA$ in the formulae for the $C$'s replaced by $\rho = b+iA$, (recall $\rho$ is the integral over the dual curve of the Kähler parameter: $\rho = \int_{\widetilde{E}} \omega$). This then establishes the full dictionary for the mirror map in the case of line bundles over our original curve. The map for stable vector bundles of rank $r$ is more complicated, but may be derived from the one for line bundles. This will be sketched int the next section.

### A sketch of the proof of mirror symmetry

Polishchuk and Zaslow prove categorical mirror symmetry in an entirely constructive way: they literally build the functor $\Phi_q$ between $\mathcal{D}^b(E_q)$ and $\mathcal{F}^0\left(\widetilde{E}^q\right)$. This is done in five steps:

1. The functor is defined on a certain nice vector bundles.
2. It is then checked that on this subclass the composition law is respected.
3. The functor is then shown to commute with isogenies.
4. Using this, it is extended to all vector bundles.
5. Finally, the functor is defined for "skyscraper-like" sheaves.

Each of these steps will now be discussed in turn.

*Step 1*

Here the functor is constructed for the full subcategory of vector bundles of the form $L(\phi)\otimes F(V, \exp N)$ (henceforth referred to as $\mathcal{L}(E_q)$), where $\phi = t_x^*\phi_0 \cdot \phi_0^{n-1}$, for some $x \in E_q$, $n \in \mathbb{Z}$, with the notation as in Sec. . The map on objects is

$$\Phi_q : (L(t^*_{\alpha\tau+\beta}\phi_0 \cdot \phi_0^{n-1}) \otimes F(V, \exp N)) \mapsto (\Lambda,\, A),$$

where $\Lambda$ is the line (parameterised by $t$)

$$\Lambda = (\alpha + t,\, (n-1)\alpha + nt),$$

and $d + A$ is the connection on the local system, with

$$A = (-2\pi i\beta \cdot \mathbf{1}_V + N)dx.$$

Thus, as illustrated in the previous section, bundles of degree $n$ get mapped to lines with slope $n$. Also, translation on the Jacobian has two markedly different effects: it translates the line on Fukaya side, and changes the monodromy on the local system.

The construction of the map on morphisms is somewhat more complicated, but no more technical. The functor needs to be a map

$$\Phi_q : \mathrm{Hom}\left(L(\phi_1)\otimes F(V_1, \exp N_1,\, L(\phi_2)\otimes F(V_2, \exp N_2\right) \to \mathrm{Hom}^0\left(\Phi_q(L(\phi_1)\otimes F(V_1, \exp N_1),\, \Phi_q(L(\phi_2)\otimes F(V_2, \exp N_2)\right).$$

Prop. .2 is used to understand the LHS of this map. The only interesting case is when $n_1 < n_2$ as there are no homomorphisms when $n_1 > n_2$, and the problem reduces to homomophisms on vector spaces when $n_1 = n_2$. In the interesting case, then, the proposition shows

$$LHS = H^0(L(\phi_2\phi_1^{-1})) \otimes (V_1^* \otimes V_2).$$

The RHS, on the other hand, is simply

$$RHS = \bigoplus_{e_k \in \Lambda_1\cap\Lambda_2} V_1^* \otimes V_2 \cdot e_k.$$

With some work the functor may then be explicitly defined. Indeed, defining

$$\alpha_{12} = \frac{\alpha_2 - \alpha_1}{n_2 - n_1}, \quad \beta_{12} = \frac{\beta_2 - \beta_1}{n_2 - n_1},$$

and writing

$$f_k = \theta\left[\frac{k}{n_2 - n_1}, 0\right]((n_2 - n_1)\tau,\, (n_2 - n_1)(z + \alpha_{12} + \beta_{12})),$$

then for any $T \in V_1^* \otimes V_2$, we have

$$\begin{aligned}\Phi_q(\mathcal{V}(f_k \otimes T)) = &\exp(-i\pi\tau\alpha_{12}^2(n_2 - n_1)) \\ &\exp[\alpha_{12}(N_2 - N_1^* - 2\pi i(n_2 - n_1)\beta_{12}] \cdot T\, e_k.\end{aligned}$$

Note that the $f_k$ are a standard basis for the theta functions in $H^0(L(\phi_2\phi_1^{-1}))$. Thus it bears out the observations made in the previous section, that $\alpha_{12}$ and $\beta_{12}$ effect shifts and monodromies on the RHS of the functor, which determines the identification up to a constant.

*Step 2*

The functor must now be checked to respect composition in both categories. That is to say, we need

$$\Phi_q \circ m_2 = m_2 \circ \Phi_q.$$

This involves a great deal of computation, but none of it is too hard. The principal tools needed to understand composition of sections on the derived category side are Prop. .3 and the addition formula for theta functions. Simple plane geometry is needed to understand the Fukaya side. Doing this carefully, and bearing in mind Fukaya's prescription for computing $m_2$, one obtains the fact that the functor does indeed respect composition. In essence, this part of the proof is very similar in flavour to the examples in the previous section where we computed what Fukaya's prescription told us about composition, and saw that we came up with the addition formula for theta functions.

*Step 3*

This step examines the behaviour of $\Phi_q$ under isogeny. Vector bundles pull-back and push-forward under isogenies: and one may ask what the corresponding behaviour on the Fukaya side is. The map $\pi_r : \mathbb{R}^2/\mathbb{Z}^2 \to \pi_r : \mathbb{R}^2/\mathbb{Z}^2$ which sends $(x,\, y)$ to $(rx,\, y)$ is an $r$-fold cover, and, if one sends the Kähler parameter $\rho$ to $r\rho$, it respects the Kähler structure. This then defines functors $\pi_{r*}$ and $\pi_r^*$ on between Fukaya categories. It is then straightforward to check:

$$\begin{array}{ccc} \mathcal{L}\left(E_q\right) & \xrightarrow{\Phi_q} & \mathcal{F}\left(\widetilde{E}^{\rho}\right) \\ \downarrow{\scriptstyle \pi_r^*} & & \downarrow{\scriptstyle \pi_r^*} \\ \mathcal{L}\left(E_{q^r}\right) & \xrightarrow{\Phi_{q^r}} & \mathcal{F}\left(\widetilde{E}^{r\rho}\right) \end{array} .$$

*Step 4*

Using the Prop. .1 and step three, it is now easy to extend $\Phi_q$ to all vector bundles. To extend it to all homomorphisms between vector bundles, Prop. .2 is again used.

*Step 5*

All that remains at this point is to define $\Phi_q$ on "skyscraper-like" sheaves. It should not be a surprise that these go to the only lines remaining, namely vertical ones. The extension is not difficult, and may again be made very concrete.

## Final remarks

While categorical mirror symmetry may seem rather abstract and esoteric in full generality, it is hoped that this discussion showed that it may be expressed very concretely when attention is restricted to elliptic curves.


## ACKNOWLEDGMENTS

We would like to thank Prof. Farkas for the rewarding classroom experience.



[1] B. R. Greene and M. R. Plesser, "Duality In Calabi-Yau Moduli Space," Nucl. Phys. B **338**, 15 (1990).

[2] D. A. Cox and S. Katz, "Mirror symmetry and algebraic geometry," SPIRES entry

[3] A. Strominger, S. T. Yau and E. Zaslow, "Mirror symmetry is T-duality," Nucl. Phys. B **479**, 243 (1996) [arXiv:hep-th/9606040].

[4] A. Polishchuk and E. Zaslow, "Categorical mirror symmetry: The Elliptic curve," Adv. Theor. Math. Phys. **2**, 443 (1998) [arXiv:math.ag/9801119].

[5] E. Witten, "Phases of N = 2 theories in two dimensions," Nucl. Phys. B **403**, 159 (1993) [arXiv:hep-th/9301042].

[6] A. Kapustin and D. Orlov, "Vertex algebras, mirror symmetry, and D-branes: The case of complex tori," Commun. Math. Phys. **233**, 79 (2003) [arXiv:hep-th/0010293].

[7] A. Kapustin and D. Orlov, "Lectures on mirror symmetry, derived categories, and D-branes," arXiv:math.ag/0308173.

[8] M. Kontsevich, "Rigorous results in topological sigma model," SPIRES entry *Prepared for 11th International Conference on Mathematical Physics (ICMP-11) (Satellite colloquia: New Problems in the General Theory of Fields and Particles, Paris, France, 25-28 Jul 1994), Paris, France, 18-23 Jul 1994*

[9] J. Distler, H. Jockers and H. j. Park, "D-brane monodromies, derived categories and boundary linear sigma models," arXiv:hep-th/0206242.