$A-$discriminants for the reflexive polygons

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1 Introduction

Given a polynomial $f \in \mathbb{C}[x, y]$ we want to determine whether it has a singularity or not. In order to do this, we first focus on the monomials of the polynomial: let's call A the set of those monomials. The $A-$*discriminant* Δ_A is the unique polynomial satisfying

$$\begin{cases} \textit{irreducible over } \mathbb{Z} \\ \textit{homogeneous in the coefficients of } f \\ \textit{vanishes when } f \textit{ has a singularity} \end{cases}$$

Let's make clear what the last condition means: Δ_A is a polynomial in as many variables as monomials A has. Δ_A will have a root α iff the polynomial f whose coefficients are given by the coordinates of α has a singularity.

Any integer point in the plane can be regarded as the multidegree of a monomial in two variables. For instance,the point $(3, 2)$ can be thought as the monomial x^3y^2, the point $(0, 3)$ as y^3 and so on. Furthermore, given a bivariate polynomial we can disregard the coefficients and look at the monomials, and consider the convex hull of the points they define in the plane. This convex hull is a polygon, called the *polytope* of the polynomial. Given a polygon in the plane, we say that it is *reflexive* iff it has exactly one interior integer point.

Poonen and Rodriguez-Villegas classified in [2] the 16 equivalence classes of reflexive polygons (FIG.1).

Our goal is to find the $A-$discriminant for the families of bivariate polynomials defined by each reflexive polygon.

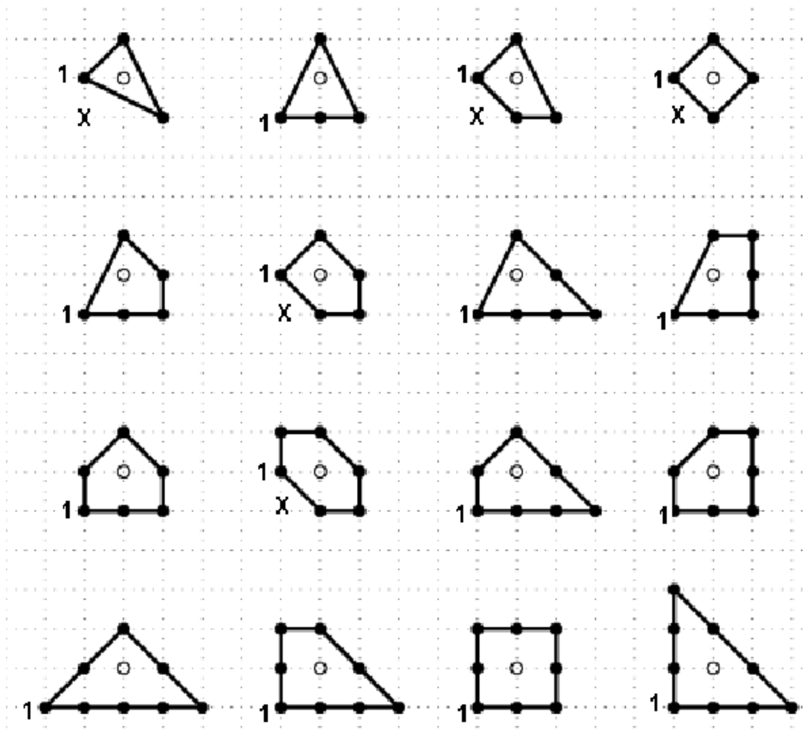

Figure 1: reflexive polygons

Let's name the set of monomials of each polytope:

$$\begin{aligned}
A_1 &= \{(2,0),(0,1),(1,1),(1,2)\} \\
A_2 &= \{(0,0),(1,0),(2,0),(1,1),(1,2)\} \\
A_3 &= \{(1,0),(2,0),(0,1),(1,1),(1,2)\} \\
A_4 &= \{(1,0),(0,1),(1,1),(2,1),(1,2)\} \\
A_5 &= \{(0,0),(1,0),(2,0),(1,1),(2,1),(1,2)\} \\
A_6 &= \{(1,0),(2,0),(0,1),(1,1),(2,1),(1,2)\} \\
A_7 &= \{(0,0),(1,0),(2,0),(3,0),(1,1),(2,1),(1,2)\} \\
A_8 &= \{(0,0),(1,0),(2,0),(1,1),(2,1),(1,2),(2,2)\} \\
A_9 &= \{(0,0),(1,0),(2,0),(0,1),(1,1),(2,1),(1,2)\}
\end{aligned}$$

$$
\begin{aligned}
A_{10} &= \{(1,0),(2,0),(0,1),(1,1),(2,1),(0,2),(1,2)\} \\
A_{11} &= \{(0,0),(1,0),(2,0),(3,0),(0,1),(1,1),(2,1),(1,2)\} \\
A_{12} &= \{(0,0),(1,0),(2,0),(0,1),(1,1),(2,1),(1,2),(2,2)\} \\
A_{13} &= \{(0,0),(1,0),(2,0),(3,0),(4,0),(1,1),(2,1),(3,1),(2,2)\} \\
A_{14} &= \{(0,0),(1,0),(2,0),(3,0),(0,1),(1,1),(2,1),(0,2),(1,2)\} \\
A_{15} &= \{(0,0),(1,0),(2,0),(0,1),(1,1),(2,1),(0,2),(1,2),(2,2)\} \\
A_{16} &= \{(0,0),(1,0),(2,0),(3,0),(0,1),(1,1),(2,1),(0,2),(1,2),(0,3)\}
\end{aligned}
$$

The correspondant polynomials (whose coefficients are indeterminates) are:

$$
\begin{aligned}
f_1 &= a_{20}x^2 + a_{01}y + a_{11}xy + a_{12}xy^2 \\
f_2 &= a_{00} + a_{10}x + a_{20}x^2 + a_{11}xy + a_{12}xy^2 \\
f_3 &= a_{10}x + a_{20}x^2 + a_{01}y + a_{11}xy + a_{12}xy^2 \\
f_4 &= a_{10}x + a_{01}y + a_{11}xy + a_{21}x^2y + a_{12}xy^2 \\
f_5 &= a_{00} + a_{10}x + a_{20}x^2 + a_{11}xy + a_{21}x^2y + a_{12}xy^2 \\
f_6 &= a_{10}x + a_{20}x^2 + a_{01}y + a_{11}xy + a_{21}x^2y + a_{12}xy^2 \\
f_7 &= a_{00} + a_{10}x + a_{20}x^2 + a_{30}x^3 + a_{11}xy + a_{21}x^2y + a_{12}xy^2 \\
f_8 &= a_{00} + a_{10}x + a_{20}x^2 + a_{11}xy + a_{21}x^2y + a_{12}xy^2 + a_{22}x^2y^2 \\
f_9 &= a_{00} + a_{10}x + a_{20}x^2 + a_{01}y + a_{11}xy + a_{21}x^2y + a_{12}xy^2 \\
f_{10} &= a_{10}x + a_{20}x^2 + a_{01}y + a_{11}xy + a_{21}x^2y + a_{02}y^2 + a_{12}xy^2 \\
f_{11} &= a_{00} + a_{10}x + a_{20}x^2 + a_{30}x^3 + a_{01}y + a_{11}xy + a_{21}x^2y + a_{12}xy^2 \\
f_{12} &= a_{00} + a_{10}x + a_{20}x^2 + a_{01}y + a_{11}xy + a_{21}x^2y + a_{12}xy^2 + a_{22}x^2y^2 \\
f_{13} &= a_{00} + a_{10}x + a_{20}x^2 + a_{30}x^3 + a_{40}x^4 + a_{11}xy + a_{21}x^2y + a_{31}x^3y + a_{22}x^2y^2 \\
f_{14} &= a_{00} + a_{10}x + a_{20}x^2 + a_{30}x^3 + a_{01}y + a_{11}xy + a_{21}x^2y + a_{02}y^2 + a_{12}xy^2 \\
f_{15} &= a_{00} + a_{10}x + a_{20}x^2 + a_{01}y + a_{11}xy + a_{21}x^2y + a_{02}y^2 + a_{12}xy^2 + a_{22}x^2y^2 \\
f_{16} &= a_{00} + a_{10}x + a_{20}x^2 + a_{30}x^3 + a_{01}y + a_{11}xy + a_{21}x^2y + a_{02}y^2 + a_{12}xy^2 + a_{03}y^3
\end{aligned}
$$

Note that between all these 16 polynomials, we have two different categories: cubics and quartics. The very last one, f_{16} is the general cubic. The quartics we have are $f_8, f_{12}, f_{13}, f_{15}$.

It might therefore be a good idea to try to solve first the general cubic, and then analyse how that applies to the other cubics just by replacing the correspondent coefficients with zeroes. Later we'll take care of the quartics.

2 Methods

So let's start with the general cubic:

$$f_{16} = a_{00} + a_{10}x + a_{20}x^2 + a_{30}x^3 + a_{01}y + a_{11}xy + a_{21}x^2y + a_{02}y^2 + a_{12}xy^2 + a_{03}y^3$$

$$A_{16} = \{(0,0),(1,0),(2,0),(3,0),(0,1),(1,1),(2,1),(0,2),(1,2),(0,3)\}$$

To compute the A-discriminant, we first homogeneize the original polynomial f_{16} :

$$a_{003}z^3 + a_{102}xz^2 + a_{201}x^2z + a_{300}x^3 + a_{012}yz^2 + a_{111}xyz + a_{210}x^2y + a_{021}y^2z + a_{120}xy^2 + a_{030}y^3$$

We will abuse notation and keep calling f_{16} the homogeneized polynomial. The following program in Maple ia a courtesy of Bernd Sturmfels[1]:

```
> interface(quiet=true):
> Cubic :=  a300 * x^3 +  a210 * x^2*y +  a201 * x^2*z +
>           a120 * x*y^2 +  a111 * x*y*z +  a102 * x*z^2 +
>           a030 * y^3 +  a021 * y^2*z +  a012 * y*z^2 +  a003 * z^3 ;
>
> f := diff(Cubic,x): g := diff(Cubic,y): h := diff(Cubic,z):
>
> with(linalg): J := det(jacobian([f,g,h],[x,y,z])):
>
> resmat := []:
> for f_{16}q in [f,g,h,diff(J,x),diff(J,y),diff(J,z)] do
> resmat := [resmat[], [coeff(q,x,2), coeff(q,y,2), coeff(q,z,2),
> coeff(coeff(q,x,1),y,1),coeff(coeff(q,x,1),z,1),coeff(coeff(q,y,1),z,1)]]:
> od:
>
> discriminant := sort(det(array(resmat))/13824,
> [a300,a210,a201,a120,a111,a102,a030,a021,a012,a003]):
>
> lprint(discriminant);
> degree(discriminant), nops(discriminant);
```

Maple gives us the A-discriminant of the general cubic, which is a polynomial of degree 12 in 10 variables (homogeneous and irreducible) with 2040 monomials: `http://www.ma.utexas.edu/users/sadduci/sturmfels.mw` .

[1]University of California Berkley

3 Cubics

Now let's use this information to compute the A-discriminants of the reminder cubics.

3.1 $f_1 = a_{20}x^2 + a_{01}y + a_{11}xy + a_{12}xy^2$

Following the same procedure, we first homogeneize it
$f_1 = a_{201}x^2z + a_{012}yz^2 + a_{111}xyz + a_{12z}xy^2$
Now, we substitute in f_{16} all of the variables not in f_1 by zero, namely:

```
> f1:=subs(a300=0, a210=0, a102=0, a030=0, a021=0, a003=0, Cubic);
```

and we do the same thing in the A-discriminant:

```
d1:=subs(a300=0, a210=0, a102=0, a030=0, a021=0, a003=0, d0);
```

We obtain:

$$d1 = -a_{120}^3 a_{201}^3 a_{012}^3 (27a_{201}a_{120}a_{012} + a_{111}^3)$$

We know that $\Delta_{A_1} \mid d_1$ and that Δ_{A_1} is not a monomial, so it has to be [2] $\Delta_{A_1} = 27a_{20}a_{12}a_{01} + a_{11}^3$.

So let's check that our conclusion is correct. Let's call $d2 = 27a_{201}a_{120}a_{012} + a_{111}^3$

In order to make clear the definition of the discriminant as describing the coefficients describing hypersurfaces in the family with singular points we choose an arbitrary substitution

```
> d3:=subs(a201=1, a120=1, a012=1, d2);
                       d3 := -27 - a111^3
> roots(d3);
                            [[-3, 1]]
```

[2]Note that we cannot decide whether $\Delta_{A_1} = 27a_{20}a_{12}a_{01} + a_{11}^3$ or $\Delta_{A_1} = -27a_{20}a_{12}a_{01} - a_{11}^3$.

This means that the point with $\begin{cases} a_{201} & = & 1 \\ a_{120} & = & 1 \\ a_{012} & = & 1 \\ a_{111} & = & -3 \end{cases}$ is a root of the A-discriminant.

We substitute these values in f_1 and obtain:

```
> f1:=subs(a201=1, a120=1, a012=1, a111=-3, f1);
               f1 := x^2 z + y z^2 - 3 x y z + x y^2
> f1:=subs(z=1, f1);
               f1 := x^2   + y  - 3 x y  + x y^2
```

So with our arbitrary substitution we get the polynomial:

$$f_1 = x^2 + y - 3xy + xy^2$$

Let's see that f_1 has a *multiple root*, *ie*, $\exists$ a point (x, y)[3] such that $\begin{cases} f_1(x,y) & = & 0 \\ \dfrac{\partial f_1}{\partial x}(x,y) & = & 0 \\ \dfrac{\partial f_1}{\partial y}(x,y) & = & 0 \end{cases}$

One way to solve this system of polynomial equations is using resultants.

We first compute the resultants of f_1 and $\frac{\partial f_1}{\partial x}$ and of f_1 and $\frac{\partial f_1}{\partial y}$ with respect to the variable y. We obtain two polynomials on the variable x.

Next, we compute the roots of these two polynomials, and substitute the common roots (other that zero) in f_1. Then we compute the roots (in the variable y) of the new f_1 and check if the final roots (x, y) (for $y \neq 0$) are also roots of $\frac{\partial f_1}{\partial x}$ and $\frac{\partial f_1}{\partial y}$. This is the Maple file:

```
> fx:=diff(f1,x);  fy:=diff(f1,y);
                                          2
                     fx := 2 x - 3 y + y
                    fy := 1 - 3 x + 2 x y
> rx:=resultant(f1, fx, y);
                                  2     4
                    rx := 2 x - 3 x  + x
> ry:=resultant(f1, fy, y);
                          4         2      3
                 ry := 4 x  - x + 6 x  - 9 x
```

[3]Note that since we are working with the homogeneized polynomial, we are only interested in solutions in the torus $\mathbb{C}^{*2}$.

```
> roots(rx);
                   [[0, 1], [1, 2], [-2, 1]]
> roots(ry);
                    [                 [1   ]]
                    [[0, 1], [1, 2], [-, 1]]
                    [                 [4   ]]
> f2:=subs(x=1, f1);
                                     2
                      f2 := 1 - 2 y + y
> roots(f2);
                           [[1, 2]]
> subs(x=1, y=1, fx);
                              0
> subs(x=1, y=1, fy);
                              0
```

Hence $(1,1)$ is a multiple root of f_1 and the $A-$discriminant of f_1 is

$$\Delta_{A_1} \;=\; 27a_{20}a_{12}a_{01} + a_{11}^3.$$

3.2 $f_2 \;= a_{00} + a_{10}x + a_{20}x^2 + a_{11}xy + a_{12}xy^2$

Homogeneization:
$f_2 \;= a_{003}z^3 + a_{102}xz^2 + a_{201}x^2z + a_{111}xyz + a_{120}xy^2$.
We obtain that with the substitution

```
>f2 := subs(a120=1, a003=1, a201=1, a102=2, a111=4, f2);
```

the polynomial $f_2 \;=\; 1 + 2x + x^2 + 4xy + xy^2$ has a multiple root at $(1, -2)$. Hence, the A-discriminant given by Maple by the same procedure as before is the following degree 4 homogeneous polynomial:

$$\Delta_{A_2} \;=\; -64a_{12}^2a_{00}a_{20} + 16a_{10}^2a_{12}^2 - 8a_{10}a_{11}^2a_{12} + a_{11}^4.$$

3.3 $f_3 \;= a_{10}x + a_{20}x^2 + a_{01}y + a_{11}xy + a_{12}xy^2$

Homogeneization:
$f_3 \;= a_{102}xz^2 + a_{201}x^2z + a_{012}yz^2 + a_{111}xyz + a_{120}xy^2$.
This time, with the substitution

```
>f3 := subs(a120=a201, a201=a111, a111=16, a102=a012, a012=-1, f3);
```

the polynomial $f_3 = -x+16x^2-y+16xy+16xy^2$ has a multiple root at $(1/8,-1/4)$. Hence the A-discriminant given by Maple by the same procedure as before is the following degree 5 homogeneous polynomial:

$$\Delta_{A_3} = 16a_{12}^2a_{10}^3 + a_{11}^4a_{10} - 8a_{12}a_{11}^2a_{10}^2 + 36a_{12}a_{01}a_{20}a_{11}a_{10} - a_{01}a_{20}a_{11}^3 - 27a_{12}a_{01}^2a_{20}^2.$$

3.4 $f_4 = a_{102}xz^2 + a_{012}yz^2 + a_{111}xyz + a_{210}x^2y + a_{120}xy^2.$

Substitution:

```
>f4 := subs(a111=a102, a102=a120, a120=a210, a210=4, a012=1, f4);
```

$f_4 = 4x+y+4xy+4x^2y+4xy^2$ has a mutliple root at $(1/2,-1)$. The A-discriminant of f_4 is has degree 4:

$$\Delta_{A_4} = -8a_{21}a_{11}^2a_{01} + a_{11}^4 + 16a_{12}^2a_{10}^2 - 8a_{12}a_{10}a_{11}^2 - 32a_{21}a_{12}a_{10}a_{01} + 16a_{21}^2a_{01}^2.$$

3.5 $f_5 = a_{003}z^3 + a_{102}xz^2 + a_{201}x^2z + a_{111}xyz + a_{210}x^2yz + a_{120s}xy^2.$

Substitution:

```
>f5 := subs(a210=1, a201=-1, a102=1, a120=2, a111=1, a003=-4/27, f5);
```

$f_5 := -\frac{4}{27}+x-x^2+xy+x^2y+2xy^2$ has a mutliple root at $(\frac{1}{3},-\frac{1}{3})$. The A-discriminant of f_5 is the following homogeneus polynomial of degree 7 with 16 monomials:

$\Delta_{A_5} = 16a_{12}^3a_{20}^2a_{10}^2+8a_{12}a_{21}a_{20}a_{11}^3a_{10}+96a_{00}a_{12}^2a_{21}a_{20}^2a_{11}+a_{12}a_{20}^2a_{11}^4+16a_{12}^2a_{21}^2a_{10}^3- 16a_{12}^2a_{21}a_{20}a_{11}a_{10}^2 - 27a_{00}^2a_{12}a_{21}^4 - 8a_{12}a_{21}^2a_{11}^2a_{10}^2 - 30a_{00}a_{12}a_{21}^2a_{20}a_{11}^2 + a_{21}^2a_{11}^4a_{10} - 64a_{00}a_{12}^3a_{20}^3+36a_{00}a_{12}a_{21}^3a_{11}a_{10}-a_{21}a_{20}a_{11}^5-8a_{12}^2a_{20}^2a_{11}^2a_{10}-a_{00}a_{21}^3a_{11}^3-72a_{00}a_{12}^2a_{21}^2a_{20}a_{10}$

3.6 $f_6 = a_{102}xz^2 + a_{201}x^2z + a_{012}yz^2 + a_{111}xyz + a_{210}x^2y + a_{120}xy^2$

Substitution:

```
>f6 := subs(a120=1, a210=-1, a102=-1, a201=-2, a111=2,
       a003=-1, a012=1, f6);
```

$f_6 = x-2x^2+y+2xy-x^2y+xy^2$ has a mutliple root at $(-1,-1)$. The A-discriminant of f_6 is the following homogeneus polynomial of degree 8 with 25 monomials:

$\Delta_{A_6} = 16*a120^2*a210^2*a102^4+16*a120^3*a201^2*a102^3-27*a120^2*a012^2*a201^4-24*a120*a201^2*a210^2*a012^2*a102-a210*a201*a111^5*a102-16*a120^2*a210*a201*a111*a102^3+36*a120^2*a012*a201^3*a111*a102-8*a012*a210^3*a111^2*a102^2-24*a120^2*a012*a210*a201^2*a102^2+8*a120*a210*a201*a111^3*a102^2+36*a120*a012^2*a210*a201^3*a111-8*a120^2*a201^2*a111^2*a102^2+16*a012^2*a210^4*a102^2-8*a012^2*a210^2*a201^2*a111^2+a012*a210*a201^2*a111^4-8*a120*a210^2*a111^2*a102^3+a210^2*a111^4*a102^2-a120*a012*a201^3*a111^3-32*a120*a012*a210^3*a102^3+16*a201^2*a210^3*a012^3+a120*a201^2*a111^4*a102+64*a120*a012*a210^2*a201*a111*a102^2-46*a120*a012*a210*a201^2*a111^2*a102+8*a012*a210^2*a201*a111^3*a102-16*a210^3*a111*a201*a012^2*a102$

3.7 $f_7 = a_{003}z^3 + a_{102}xz^2 + a_{201}x^2z + a_{300}x^3 + a_{111}xyz + a_{210}x^2y + a_{120}xy^2$

Substitution:

```
>f7:=subs(a111=3, a210=-1, a003=-1, a102=-1, a300=-1,
          a120=-1, a201=2, f7);
```

$f_7 = -1-x+2x^2-x^3+3xy-x^2y-xy^2$ has a mutliple root at $(1,1)$. The A-discriminant of f_7 is the following homogeneus polynomial of degree 7 in 26 monomials:

$\Delta_{A_7} = a300*a111^6-a003*a210^3*a111^3-27*a003^2*a120*a210^4-64*a003*a120^3*a201^3-432*a003^2*a120^3*a300^2+16*a120^2*a210^2*a102^3-64*a120^3*a300*a102^3+a120*a201^2*a111^4+216*a003^2*a120^2*a300*a210^2-a210*a201*a111^5+a210^2*a111^4*a102+16*a120^3*a201^2*a102^2-16*a120^2*a210*a201*a111*a102^2+48*a120^2*a300*a111^2*a102^2-12*a120*a300*a111^4*a102+36*a003*a120*a210^3*a111*a102-72*a003*a120^2*a210^2*a201*a102-72*a003*a120^2*a300*a201*a111^2-144*a003*a120^2*a300*a210*a111*a102-8*a120^2*a201^2*a111^2*a102+288*a003*a120^3*a300*a201*a102+36*a003*a120*a300*a210*a111^3-30*a003*a120*a210^2*a201*a111^2+96*a003*a120^2*a210*a201^2*a111-8*a120*a210^2*a111^2*a102^2+8*a120*a210*a201*a111^3*a102$

3.8 $f_9 = a_{003}z^3 + a_{102}xz^2 + a_{201}x^2z + a_{012}yz^2 + a_{111}xyz + a_{210}x^2y + a_{120}xy^2$

Substitution:

```
>f9:=subs(a111=a210, a210=a003, a102=1, a003=a300,
          a300=a120, a120=a201, a201=a012, a012=4, f9);
```

$f_9 = 4 + x + 4x^2 + 4y + 4xy + 4x^2y + 4xy^2$ has a mutliple root at $(1, -3/2)$. The A-discriminant of f_9 is the following homogeneus polynomial of degree 10 in 80 monomials:

Δ_{A_9} = `http://www.ma.utexas.edu/users/sadduci/f9sturmfels.mw`

3.9 $f_{10} = a_{102}xz^2 + a_{201}x^2z + a_{012}yz^2 + a_{111}xyz + a_{210}x^2y + a_{021}y^2z + a_{120}xy^2$

In this case, the discriminant of the polynomial is already irreducible, hence it has to be the A-discriminant (since the A-discriminant is a factor of the discriminant). It has degree 12 and 127 monomials.

$\Delta_{A_{10}}$ = `http://www.ma.utexas.edu/users/sadduci/f10sturmfels.mw`.

3.10 $f_{11} = a_{003}z^3 + a_{102}xz^2 + a_{201}x^2z + a_{300}x^3 + a_{012}yz^2 + a_{111}xyz + a_{210}x^2y + a_{120}xy^2$

$\Delta_{A_{11}}$ is a polynomial of degree 10 in 154 monomials:

`http://www.ma.utexas.edu/users/sadduci/f11sturmfels.mw`.

3.11 $f_{14} = a_{003}z^3 + a_{102}xz^2 + a_{201}x^2z + a_{300}x^3 + a_{012}yz^2 + a_{111}xyz + a_{210}x^2y + a_{021}y^2z + a_{120}xy^2$

$\Delta_{A_{14}}$ is a polynomial of degree 12 in 774 monomials:

`http://www.ma.utexas.edu/users/sadduci/f11sturmfels.mw`.

4 Quartics

We will use the following program in Singular (courtesy of Alicia Dickenstein[4]).

We then declare a ring with variables $x1, ..., xn$ for the coefficients, the 2 variables $t1, t2$, ($t1$ will play the role of our variable x, and $t2$ of y); and an auxiliary variable u. We call this ring r. The "dp" at the end of the first instruction stands for a term order that Singular requires.

Then we define our ideal, called i, generated by f, its partials with respect to $t1$ and $t2$, and the auxiliary polynomial $g = t1t2u - 1$, whose vanishing ensures that $(t1, t2)$ lies in the torus (i.e. $t1, t2$, are both non-zero).

Then, the $A-$discriminant that we look for is the equation of the closure of the projection onto the x variables of those tuples $(x1, ..., xn, t1, t2)$ such that $f = 0$, with $(t1, t2)$ in the torus. This is computed by computing the generator of the intersection of the ideal i with the subring of the variables x (so we need to eliminate the variables $t1, t2, u$).

Let's see, as an example, how would this work for the general quartic.

$$\begin{array}{llllllllll} f := & a_{00} & + & a_{10}x & + & a_{20}x^2 & + & a_{30}x^3 & + & a_{40}x^4 & + \\ & a_{01}y & + & a_{11}xy & + & a_{21}x^2y & + & a_{31}x^3y & + \\ & a_{02}y^2 & + & a_{12}xy^2 & + & a_{22}x^2y^2 & + \\ & a_{03}y^3 & + & a_{13}xy^3 & + \\ & a_{04}y^4 \end{array}$$

Here we have 15 coefficients, so $n = 15$.

```
> ring r = 0,(x1,x2,x3,x4,x5,x6,x7,x8,x9,x10,x11,x12,x13,x14,x15,
             t1,t2,u), dp;
> poly f = x1+x2*t1+x3*t1^2+x4*t1^3+x5*t1^4+
          x6*t2+x7*t1*t2+x8*t1^2*t2+x9*t1^3*t2+
  x10*t2^2+x11*t1*t2^2+x12*t1^2*t2^2+
  x13*t2^3+x14*t1*t2^3+
  x15*t2^4;
> poly f1 = diff(f,t1);
> poly f2 = diff(f,t2);
> poly g = t1*t2*u-1;
```

[4]University of Buenos Aires, Argentina.

```
> ideal i = f,f1,f2,g;
> ideal k = eliminate(i, t1*t2*u);
> k;
```

The computer got stuck and couldn't compute this. So let's keep trying for the polynomials we actually need.

4.1 $f_8 = a_{00} + a_{10}x + a_{20}x^2 + a_{11}xy + a_{21}x^2y + a_{12}xy^2 + a_{22}x^2y^2$

```
> ring r = 0,(x1,x2,x3,x4,x5,x6,x7,t1,t2,u), dp;
> poly f = x1+ x2*t1 + x3*t1^2+ x4*t1*t2+x5*t1^2*t2
  +x6*t1*t2^2+x7*t1^2*t2^2;
> poly f1 = diff(f,t1);
> poly f2 = diff(f, t2);
> poly g =t1*t2*u-1;
> ideal i = f,f1,f2,g;
> ideal k = eliminate(i, t1*t2*u);
> k;
k[1]=x3*x4^5*x5*x6-x2*x4^4*x5^2*x6+x1*x4^3*x5^3*x6
-x3^2*x4^4*x6^2-8*x2*x3*x4^3*x5*x6^2
+8*x2^2*x4^2*x5^2*x6^2+30*x1*x3*x4^2*x5^2*x6^2
-36*x1*x2*x4*x5^3*x6^2+27*x1^2*x5^4*x6^2
+8*x2*x3^2*x4^2*x6^3+16*x2^2*x3*x4*x5*x6^3
-96*x1*x3^2*x4*x5*x6^3-16*x2^3*x5^2*x6^3
+72*x1*x2*x3*x5^2*x6^3-16*x2^2*x3^2*x6^4
+64*x1*x3^3*x6^4-x3*x4^6*x7+x2*x4^5*x5*x7
-x1*x4^4*x5^2*x7+10*x2*x3*x4^4*x6*x7-8*x2^2*x4^3*x5*x6*x7
-44*x1*x3*x4^3*x5*x6*x7+46*x1*x2*x4^2*x5^2*x6*x7
-36*x1^2*x4*x5^3*x6*x7-32*x2^2*x3*x4^2*x6^2*x7
+80*x1*x3^2*x4^2*x6^2*x7+16*x2^3*x4*x5*x6^2*x7
-16*x1*x2*x3*x4*x5*x6^2*x7+24*x1*x2^2*x5^2*x6^2*x7
-144*x1^2*x3*x5^2*x6^2*x7+32*x2^3*x3*x6^3*x7
-128*x1*x2*x3^2*x6^3*x7-x2^2*x4^4*x7^2
+12*x1*x3*x4^4*x7^2-8*x1*x2*x4^3*x5*x7^2+8*x1^2*x4^2*x5^2*x7^2
+8*x2^3*x4^2*x6*x7^2-8*x1*x2*x3*x4^2*x6*x7^2
-64*x1*x2^2*x4*x5*x6*x7^2+160*x1^2*x3*x4*x5*x6*x7^2
+24*x1^2*x2*x5^2*x6*x7^2-16*x2^4*x6^2*x7^2
+32*x1*x2^2*x3*x6^2*x7^2+128*x1^2*x3^2*x6^2*x7^2
+8*x1*x2^2*x4^2*x7^3-48*x1^2*x3*x4^2*x7^3+16*x1^2*x2*x4*x5*x7^3
-16*x1^3*x5^2*x7^3+32*x1*x2^3*x6*x7^3-128*x1^2*x2*x3*x6*x7^3
-16*x1^2*x2^2*x7^4+64*x1^3*x3*x7^4
>
```

4.2 $f_{12} = a_{004}z^4 + a_{103}xz^3 + a_{202}x^2z^2 + a_{013}yz^3 + a_{112}xyz^2 + a_{211}x^2yz + a_{121}xy^2z + a_{220}x^2y^2$

```
> ring r = 0,(x1,x2,x3,x4,x5,x6,x7,x8,t1,t2,u), dp;
> poly f = x1 + x2*t1+x3*t1^2+x4*t2+x5*t1*t2+
           x6*t1^2*t2+x7*t1*t2^2+x8*t1^2*t2^2;
> poly f1 = diff(f,t1);
> poly f2 = diff(f,t2);
> poly g = t1*t2*u-1;
> ideal i = f,f1,f2,g;
> ideal k = eliminate(i, t1*t2*u);
> k;
```

The output is a huge polynomial that you can find in
`http://www.ma.utexas.edu/users/sadduci/f12jacob.pdf`.

4.3 Two questions

For the two last quartics,

$$f_{13} = a_{00} + a_{10}x + a_{20}x^2 + a_{30}x^3 + a_{40}x^4 + a_{11}xy + a_{21}x^2y + a_{31}x^3y + a_{22}x^2y^2$$

and

$$f_{15} = a_{00} + a_{10}x + a_{20}x^2 + a_{01}y + a_{11}xy + a_{21}x^2y + a_{02}y^2 + a_{12}xy^2 + a_{22}x^2y^2$$

we could theoretically use the previous general procedure. In practice, this doesn't work neither (the computers get stucked). Since the software we have so far doesn't seem to be enough to solve this problem, we need to develop some new techniques. This was the goal of Bernd Sturmfel's team in the Arizona Winter School 2006 [5]. The group reached a conjecture on the degree of the $A-$discriminant. The work is still in process.

References

[1] I. Gelfand, M. Kapranov, A. Zelevinsky, *Determinants, Resultants, and Multidimensional Determinants* . Birkäuser, 1994.

[2] B. Poonen and F. Rodriguez-Villegas, "Lattice Polygons and the Number 12."

[5] http://swc.math.arizona.edu/oldaws/06GenlInfo.html