On Potential Density of rational Points on Abelian Varieties

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May 17th, 2006

Abstract

One says that an algebraic variety V defined over a field K has *potential density of rational points* if after some finite extension of fields L/K the set $V(L)$ is a Zariski dense set. Here we will be interested in the case where K is a number field. In the case of curves, there is a complete classification of potential density based on the genus of the given curve. In this project, our aim is to discuss what is known in the case of surfaces and what is expected for varieties in general. In particular, we are going to show that potential density holds for abelian varieties.

1 Introduction

Let K be a number field and C a curve defined over K with genus g. If $g = 0$ it is well known that C is isomorphic to $\mathbb{P}^1$, so that after a finite extension L/K the set $C(L)$ is Zariski dense in C. If $g = 1$ (the elliptic curve case) it is shown in Theorem 4.3 that potential density holds. For $g \geq 2$ Faltings proved the celebrated Mordell Conjecture showing that potential density never holds for any number field.

Since the genus is not defined for higher dimensional varieties, it is natural to search for another invariant that could be used to study the behaviour of the rational points. A natural choice for such invariant would be the Kodaira dimension, which is defined for all algebraic varieties and that, for the case of curves, can replace the above genus-based classification for the density of rational points.

For instance, a curve has genus 0 or 1 if and only if its Kodaira dimension is -1 or 0, respectively. For surfaces of Kodaira dimension -1, as one would expect, it was proved that the rational points are potentially dense. Using this analogy, it is also expected that rational points on surfaces of Kodaira dimension 0 are potentially dense, which is true for the case of abelian varieties and classes of $K3$ surfaces based on the rank of its Picard group. For surfaces of Kodaira dimension at least 1 very little is known.

2 The case of curves

As we mentioned on the introduction, potential density in the case of curves of genus g is very well known. Here we recall the cases:

1. g=0. It is trivial since we will have an isomorphism to $\mathbb{P}^1$.
2. g=1. It is a particular case of abelian variety, which is proved to be potentially dense in section 4.
3. $g \geq 2$. Potential density does not hold. This fact is due to a theorem of Faltings that states that if C is a curve of genus at least 2 defined over a number field K, then C/K is finite.

To extend this type of results (geometry $\Longrightarrow$ arithmetic) to varieties of higher dimension, one could use the notion of Kodaira dimension. To establish the relation between the genus of a curve and its Kodaira dimension, we first recall some definitions and basic results

Definition 2.1 *Let X be a smooth projective variety. For every positive integer n consider the linear system $|nK_X|$ associated to a canonical divisor K_X. If the system is empty then define the* Kodaira dimension of X *by $k(X) := -1$, otherwise let*

$$\Phi_n : X \xrightarrow{|nK_X|} \mathbb{P}^{l_n - 1}$$

be the associated rational map (well defined up to a linear change of coordinates in $\mathbb{P}^{l_n-1}$. Here $l_n = l_n(X) = \dim H^0(X, \omega_X) = \dim L(nK_X)$ are *called the* plurigenera *of* X*. The* Kodaira dimension *of* X *is the number* $k(X) := \max_{n\geq 1} \dim \Phi_n(X)$*, i.e, the maximal dimension of the image of* X *under the pluricanonical maps* Φ_n*. Notice that* $k(X) = -1$ *exactly when all the plurigenera are equal to zero.*

Let C be a projective smooth curve of genus g. Suppose W is a canonical divisor and D an arbitrary divisor, then we have

- $l(D) = deg(D) + 1 - g + l(W - D)$ (Riemann-Roch)
- $deg(W) = 2g - 2$ and $l(W) = g$
- If $deg(D) < 0$ then $l(D) = 0$
- If $m \geq n$ then $l(mD) \leq l(nD)$
- $l(0) = 1$
- If $deg(W) > 0$ then there is a $n \geq 1$ such that the map defined above associated with nW is an embedding (the divisor nW is called very ample).

Concerning density of rational points, in what follows we prove that there is no loss if we replace the genus by the Kodaira dimension. More specifically, we get the following results

1. Suppose $g = 0$ and let W be a canonical divisor. Then $deg(W) = 2g - 2 = -2$ and $deg(nW) < 0$ for all $n \geq 1$. Hence $l_n = l(nW) = 0$ for all $n \geq 1$ which implies Kodaira=-1
2. Conversely, if Kodaira=-1 we have $l_n = l(nW) = 0$ for all $n \geq 1$, in particular $l(W) = 0$ and since $g = l(W)$ we get $g = 0$.
3. Suppose $g = 1$ and let W be a canonical divisor. Then $deg(W) = 2g - 2 = 0$ and by Riemann-Roch Theorem we have $l_n = l(nW) = deg(nW)+1-g+l((W-nW) = 0+1-1+l((1-n)W) = l((1-n)W)$. But $1 = g = l(W) \geq l(nW) = l((1-n)W) \geq l(0W) = 1$ therefore $l_n = 1$ for all $n \geq 1$ and then Kodaira=0.
4. Coversely, by the first and the below arguments , if Kodaira=0, we have $g = 1$.

5. Suppose $g \geq 2$ and W is a canonical divisor.Then $deg(W) = 2g-2 \geq 2$, then there exists n such that nW is very ample. Since it gives us an embedding of the curve into $\mathbb{P}^{l_n-1}$, we have Kodaira =1.

6. Now suppose Kodaira=1, then by the first and the third arguments, we have $g \geq 2$

3 Geometry of Surfaces

An algebraic curve, up to isomorphism, is birational to a unique non-singular curve. That allows you to classify curves into birational classes using their genus. For surfaces the situation is more complicated, since, for instance, there is not a unique projective model for a given surface. A better way, as shown by Castelnuovo, Enriques and others, is to classify surfaces using the Kodaira dimension. The following table is a classical result (see [HAR])

Theorem 3.1 *Let X be a surface. Then*

k(X)	***Surface type***
-1	*Rational or ruled*
0	*Abelian, Bielliptic,K3 or Enriques*
1	*Elliptic*
2	*General type*

where

1. *A* rational surface *is a surface birational to* $\mathbb{P}^2$.

2. X *is said to be* ruled *if there exists a surjective morphism* $\pi : X \longrightarrow C$ *to a (nonsingular) curve* C*, such that the fibre* X_y *is isomorphic to* $\mathbb{P}^1$ *for all* $y \in C$ *and such that* π *admits a* section *(i.e. a morphism* $\sigma : C \longrightarrow X$ *such that* $\pi \circ \sigma = \mathrm{i}\, d_C$*).*

3. *An* abelian surface *is a complete algebraic group surface.*

4. X *is said to be* bielliptic *or* hiperlliptic *if* $X \simeq (E \times F)/G$*, where* E *and* F *are elliptic curves and* G *is a finite group of translations of* E *acting on* F *such that* $F/G \cong \mathbb{P}^1$.

5. *A $K3$ surface S is a surface where the canonical divisor class and $H^1(S, \mathcal{O}_S)$ are trivial.*

6. *An* Enriques Surface *is a surface X with zero arithmetic and geometric genus and $2K_X \equiv 0$.*

7. *X is an* elliptic surface *if it admits a fibration $\pi : X \longrightarrow C$ to a curve C such that almost all fibers of π are nonsingular elliptic curves.*

8. General type *surfaces are exactly those that do not fit into any of the previous classes, they are "the others" surfaces.*

Concernig density of Rational points, we clearly see that for surfaces with $k(X) = -1$ the analogy with curves is fulfilled, i.e. potential density holds if $k(X) = -1$. The next natural case to consider are surfaces with zero Kodaira dimension.

4 Abelian Varieties

In this section, we will prove potential density for the case of abelian varieties, and in particular we get the desired result for the case of genus 1 curves (elliptic curves) as mentioned before. We recall that abelian varieties are varieties of Kodaira dimension 0. We start with some few definitions, which will give support to our work on this section.

Let $\mathcal{A}$ be an abelian variety over a field K.

Definition 4.1 *A point $\sigma \in \mathcal{A}(K)$ is nondegenerate if the subgroup generated by σ is Zariski dense in $\mathcal{A}$.*

Definition 4.2 *The saturation Γ of $\mathcal{A}(K)$ consists of all points p in $\mathcal{A}(\overline{K})$ such that a positive multiple of p lies in $\mathcal{A}(K)$. In particular Γ contains all the torsion points.*

Proposition 4.3 *Let $\mathcal{A}$ be an abelian variety over a number field K. Then there exists a finite extension L/K such that $\mathcal{A}(L)$ contains a nondegenerate point.*

This proposition (that solves our problem for the case of abelian varieties) is proven by the following two lemmas

Lemma 4.4 *Let $\mathcal{A}$ be an abelian variety of dimension* $\dim(\mathcal{A})$ *defined over a number field K. Then there exists a finite field extension L/K such that the rank of the Mordell-Weil group $\mathcal{A}(L)$ is strictly bigger than the rank of $\mathcal{A}(K)$.*

Proof: First let us assume that $dim(\mathcal{A}) > 1$. Take a curve C of genus ≥ 2 in $\mathcal{A}$, defined over a finite extension K_1/K. By a generalization of Raynaud's theorem (Theorem F.1.1.1, pag. 434-435 on [H-S]) we have that $C \cap \Gamma$ is finite. Since C has infinitely many points in $\overline{K}$, there exists an extension L/K_1 such that $C(L)$ contains a point q outside $C \cap \Gamma$, therefore q is neither a torsion point nor a point in $A(K)$, which implies that $\mathcal{A}(L)$ has higher rank. We now consider the case $\dim(\mathcal{E}) = 1$, i.e, $\mathcal{E}$ is an elliptic curve. Write $\mathcal{A} = \mathcal{E} \times \mathcal{E}$. Then we have $\mathcal{A}(K) = \mathcal{E}(K) \times \mathcal{E}(K)$. But now,since $dim(\mathcal{A}) = 2$, we can use the previous argument and conclude that there exists a finite field extension L/K such that the rank of $\mathcal{A}(L) = \mathcal{E}(L) \times \mathcal{E}(L)$ is strictly bigger than the rank of $\mathcal{A}(K) = \mathcal{E}(K) \times \mathcal{E}(K)$, hence $\mathcal{E}(L)$ is strictly bigger than the rank of $\mathcal{E}(K)$.

Lemma 4.5 *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be abelian varieties over a number field K. Assume that $\mathcal{A}_2$ is geometrically simple and $\mathcal{A}_1$ and $\mathcal{A}_2$ have nondegenerate points p_1 and p_2. Then $\mathcal{A}_1 \times \mathcal{A}_2$ has a nondegenerate point over some finite extension L/K.*

Proof: For any pair of abelian varieties $\mathcal{A}_1$,$\mathcal{A}_2$ the group of homomorphisms $Hom(\mathcal{A}_1, \mathcal{A}_2)$ is finitely generated as a module over $\mathbb{Z}$. After a finite extension, we can assume these are all defined over K. Let

$$Hom^0(\mathcal{A}_1, \mathcal{A}_2) := Hom(\mathcal{A}_1, \mathcal{A}_2) \otimes \mathbb{Q}$$

be the group of homomorphisms defined up to isogeny. Assume that (p_1, p_2) is contained in a proper abelian subvariety $\mathcal{B} \subsetneq \mathcal{A}_1 \times \mathcal{A}_2$. Note that the projections $\pi_i|\mathcal{B}$ are surjective. Let $\mathcal{K}_1 \subset \mathcal{B}$ be the kernel of $\pi_1|\mathcal{B}$, which may be regarded as an abelian subvariety of $\mathcal{A}_2$. Counting dimensions, we have that $\mathcal{K}_1 \subsetneq \mathcal{A}_2$, hence $\mathcal{K}_1$ is finite (since $\mathcal{A}_2$ is simple). It follows that $\pi_1|\mathcal{B}$ is an isogeny and we can regard $\mathcal{B}$ as an element $\beta \in Hom^0(\mathcal{A}_1, \mathcal{A}_2)$. In particular, there exists nonzero integer d such that $(d\beta)(p_1) = dp_2$. Choose a $\mathbb{Z}$-basis $(Z_1, ..., Z_k)$ for $Hom(\mathcal{A}_1, \mathcal{A}_2)$. There exist integers $b_1, ..., b_k$, such that $(b_1 Z_1 + ... + b_k Z_k)(p_1) = dp_2$ in the Mordell-Weil group. Hence p_2 is contained in the saturation of the subgroup of $\mathcal{A}_2(K)$ generated by the images of p_1 under Z_i. Conversely, if q is not contained in this subgroup then (p_1, q) is nondegenerate. Applying lemma 4.4, we obtain a finite field extension L/K and a point $q \in \mathcal{A}_2(L)$ with the desired property.

5 Discussion on K3 surfaces

Let $\mathcal{E} \longrightarrow \mathbb{P}^1$ be an elliptic fibration. Then the generic fiber E is an elliptic curve over $K(t)$. Suppose that $\mathcal{E} \longrightarrow \mathbb{P}^1$ has a zero-section and there exists a section of infinite order in the Mordell-Weil group of $\mathcal{E}$. Then there exists a (dense) open set $U \in \mathbb{P}^1$ such that for $b \in U$ the fiber $\mathcal{E}_b$ has a point of infinite order, hence a dense set of rational points. In particular, the set of rational points in $\mathcal{E}$ is Zariski-Dense. The idea is to extend this reasoning to the case where $\mathcal{E}$ does not have a global section like the previous one. And this can be done considering multisections.

Definition 5.1 *Let $\phi : \mathcal{E} \longrightarrow B$ be an elliptic fibration. A subariety $M \subseteq \mathcal{E}$ is a* multisection of degree $d_{\mathcal{E}}(M)$ *if M is irreducible and if the degree $d_{\mathcal{E}}(M)$ of the projection $\phi : M \longrightarrow B$ is non-zero.*

A multisection is said to be *nt*-multisection (or *nontorsion*) if for a general point $b \in B$ there exist 2 points $p_b, p'_b \in M \cap \mathcal{E}_b$ such that the zero-cycle $p_b - p'_b \in \mathcal{J}(\mathcal{E}_b)$ is non-torsion.

Proposition 5.2 *If $\mathcal{E} \longrightarrow B$ is an elliptic fibration with a nt-multisection which is a rational or elliptic curve then rational points on $\mathcal{E}$ are potentially dense.*

This result suggests that one should look for rational or elliptic multisections in an elliptic fibration. This is the method used, for instance by Bogomolov and Tschinkel [B-T1] to prove the following theorem

Theorem 5.3 *Let X be an algebraic $K3$ surface with $rank(Pic(X_{\mathcal{C}})) \leq 19$ admitting a structure of an elliptic fibration. Then this fibration has infinetely many rational nt-multisections.*

Using a similar techinique they were able to prove that potential density holds also for Enriques surfaces (see [B-T3]). Another technique is to look at the automorphism group of a $K3$ surface. In that case they show

Theorem 5.4 *Let X be a $K3$ surface over a number field with an infinite group of automorphisms. Then rational points on X are potentially dense.*

The $K3$ surfaces satisfying the conditions of the 2 previous theorems are of a very special type. Indeed

Theorem 5.5 *If a $K3$ surface admits an elliptic fibration or an infinite set of automorphisms then $rank(NS(X)) \geq 2$.*

Very little is known about the density of rational points on generic $K3$ surfaces. An open problem is to find an example of a $K3$ surface with $rank(NS(X)) = 1$ and dense set of rational points.

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