Recognizing Jacobians and 2θ Functions - Brian Katz

1 Introduction

In general, Abelian varieties are hard for us to understand. Furthermore, the major source of them, Jacobians of curves, are tori with lots of extra structure. It would be nice to know when an abelian variety is the Jacobian of a curve based on an easily checked, intrinsic, geometric condition. If $\mathcal{M}_g$ is the moduli space of curves and $\mathcal{A}_g$ is the moduli space of principally polarized Abelian varieties, then we have a map $\mu : \mathcal{M}_g \to \mathcal{A}_g$, which sends a curve to its Jacobian. Torrelli's Theorem tells us that almost every Jacobian is the Jacobian of a unique curve, namely that μ is generically injective. We know, from Riemann's parameter count, that $\mathcal{M}_g$ is $(3g-3)$-dimensional. It follows that the locus of Jacobians inside $\mathcal{A}_g$ is $(3g-3)$-dimensional. On the other hand, we constructed $\mathcal{A}_g$ as an open subset of the space of symmetric $g \times g$ matrices, $\mathbb{H}_g = \{Z \in M_{g,g}(\mathbb{C}) \mid {}^tZ = Z,\ Im(Z) > 0\}$, quotiented by an action of $Sp_{2g}(\mathbb{Z})$, which is dicrete. Hence the dimension of $\mathcal{A}_g$ is the dimension of $\mathbb{H}_g$, which is $\binom{g+1}{2}$, which clearly grows faster than Riemann's linear parameter count. In other language, we are asking how to describe the locus of Jacobians inside the moduli space of principally polarized abelian varieties ($\mathcal{M}_g \subset \mathcal{A}_g$). We already knew that an elliptic curve is isomorphic to its Jacobian, so $\mathcal{M}_1 \simeq \mathcal{A}_1$, and the isomorphism is μ. Similarly, when $g = 2$ and $g = 3$, the dimensions match, and μ is again an isomrphism. When $g = 4$ the dimensions begin to differ; $\mathcal{M}_4$ sits as a hypersurface in $\mathcal{A}_4$, which happens to be degree 16. The question of describing this locus was studied by F. Schottky and is known as the Schottky problem. I will summarize two papers, one by van Geemen and van der Geer and one by Welters, that make progress towards this goal.

Throughout, C will denote a smooth, irreducible, algebraic curve, and (X, Θ), or just X, will denote a principally polarized Abelian variety, with polarization $\mathcal{L} = \mathcal{O}_X(\Theta)$, which will be assumed to be symmetric and indecomposable. We work over $\mathbb{C}$, but much of the next section is just as well stated in characteristic different from 2.

2 Conjectures

Let (X, Θ) be a principally polarized Abelian variety of dimension g. Then we know that the (basepoint free) linear system $|2\Theta|$ gives a map $\psi : X \to \mathbb{P}^{2^g-1}$ of degree 2. The image of this map is the Kummer variety of X. We proved the following in class:

Theorem 1. *If $X = \mathrm{Jac}(C)$ for some curve C, then the Kummer variety of X has a four dimensional family of 3-secant lines (when seen embedded in $\mathbb{P}^{2^g-1}$).*

That there are any 3-secant lines is unexpected, since the g-dimensional Kummer variety is sitting in a (2^g-1)-dimensional space. One might ask if this characterizes Jacobians, and partially satisfying answer has been found.

Theorem 2. *[1] If (X, Θ) is a principally polarized Abelian variety such that there exists a curve $C \subset X$ with the property that, for all $x, y, z \in C$, $\xi \in \frac{1}{2}(C - x - y - z)$, then $\psi(\xi + x)$, $\psi(\xi + y)$, and $\psi(\xi + z)$ are collinear, then $X = Jac(C)$, and $C \subset X$ is the image of the Abel-Jacobi map.*

His proof reduces to showing that C represents a certain integral cohomology class, which we previously knew to be sufficient for our Abelian variety to be a Jacobian. But he begins with a candidate curve, and finding such a curve is difficult. On the other hand, we were surprised to see any 3-secant lines. Perhaps the Kummer variety contains enough information, without the candidate C, which leads us to:

Conjecture 1 (Welters). *If (X, Θ) is a principally polarized Abelian variety whose Kummer variety has a single 3-secant line, then $X = \mathrm{Jac}(C)$ for some curve C.*

This is a much harder question because no candidate curve C is exhibited in the hypotheses. A positive proof of this conjecture has very recently been announced by Krichever but remains to be verified. Whether or not the question has been answered, more can sometimes be learned about an Abelian variety by translating a question into the language of theta functions, which we now do.

As we've discussed in class, after we pick a point on a curve, we can use the Abel map to map our curve into its Jacobian. Let C be a curve and $p \in C$ a point. Then we define $u_p : C \to \mathrm{Jac}(C)$ by $u_p(q) = q - p \in \mathrm{Pic}^0(C) \simeq \mathrm{Jac}(C)$. This is lovely in most respects, but has one glaring shortcoming to our mathematical aesthetics: we have chosen a point on the curve. We can fix this problem by allowing our point p to vary through the curve as well. We are then lead to consider the surface $C - C = \{p - q \mid p, q \in C\}$ inside the Jacobian of our curve. The major idea behind this discussion can be summarized as hoping to find an intrinsic definition of this locus for all Abelian varieties that specifies to $C - C$ in the case of a Jacobian and then hoping to use properties of this locus to determine if it is a Jacobian.

Of course, we haven't yet used the polarization on our Abelian variety. And it turns out that $C - C$ has a nice relationship to the theta divisor giving the polarization. Let $\Theta \subset \mathrm{Jac}(C)$ be a copy of the theta divisor, and denote the singular locus of Θ by $Sing(\Theta)$. Welters first proves that

$$C - C = \{L \in \mathrm{Jac}(C) \mid L + Sing(\Theta) \subset \Theta\},$$

at least for $g(C) \geq 5$. The proof is long and cohomological.

If we think of Θ sitting in $\mathrm{Pic}^{g-1}(C)$ as W^0_{g-1}, the line bundles with a section, then the singular locus is the set of line bundles with at least two sections. If we also make Θ symmetric, we'll see later that this is related to asking for sections that vanish with order 4 at the origin.

Definition 1. *Let (X, Θ) be a principally polarized Abelian variety. Let $m_\alpha(f)$ deonte the multiplicity of a function f (or divisor) at a point α. Then define $\Gamma_{00} = \{s \in \Gamma(\mathcal{O}_X(2\Theta)) \mid m_0(s) \geq 4\}$. And define*

$$F_X = \{x \in X \mid s(x) = 0 \ for\ all\ s \in \Gamma_{00}\}.$$

Note that this is just the base locus of Γ_{00}.

Conjecture 2. *[2] If $X \simeq \mathrm{Jac}(C)$ for some curve C, then $F_X = C - C$.*

This subvariety, F_X, is our candidate for the intrinsic locus above. Then one might hope that properties of F_X characterize Jacobians.

Conjecture 3 (van Geemen, van der Geer). *Let X be a principally polarized Abelian variety of dimension $g \geq 2$. Then X is a Jacobian if and only if $dim(F_X) \geq 2$, where the dimension is the maximal dimension of its irreducible components.*

First note that we can represent X as $\mathbb{C}^g/(\mathbb{Z}^g + \tau\mathbb{Z}^g)$ for τ in $\mathbb{H}_g$, the Siegel upper half space. Then, given τ, we have the canonical basis for $\Gamma(X, 2\Theta)$: $\{\theta_2[\sigma](\tau, z)\}_{\sigma\in(\mathbb{F}_2)^g}$. Recall that a basis for $|2\Theta|$ induces a map of degree 2:

$$Th_X : X \to \mathbb{P}^{2^g-1}.$$

This is really what we mean by ψ and the Kummer variety above. If we instead use $\theta_2[\sigma](\tau, 0)$ as τ varies, we can get a map of the moduli space, $\mathcal{A}_g(2, 4)$, to the same projective space. However, these theta functions are not well defined up to the action of $Sp_{2g}(\mathbb{Z})$ on $\mathbb{H}_g$. We resolve this by taking the smallest etale covering where the 2Θ functions are well defined and call it $\mathcal{A}_g(2, 4)$. Then we have $Th : \mathcal{A}_g(2, 4) \to \mathbb{P}^{2^g-1}$. We can hope to use these maps to the same projective space, and consider their intersections.

Conjecture 4. *[2] Let $X = \mathrm{Jac}(C)$. Then*

$$Th_X(X) \cap Th(\bar{\mathcal{A}}_g(2, 4)) = Th_X(\{\alpha \in X \mid 4\alpha \in C - C\}),$$

where $\bar{\mathcal{A}}_g(2, 4)$ means the Satake compactification of the moduli space.

Similarly to the first conjecture, one might hope that this intersection classifies Jacobians.

Conjecture 5. *[2] Let X be an indecomposable, principally polarized, Abelian variety of dimension $g \geq 2$. Then X is a Jacobian if and only if $dim(Th_X(X) \cap Th(\bar{\mathcal{A}}_g(2, 4))) \geq 2$.*

3 Results

Just a few months after van Geemen and van der Geer submitted the paper which puts forth the above conjectures, Welters announced a complete solution to the second conjecture.

Theorem 3. *[3] Let $X = \mathrm{Jac}(C)$ for $g(C) \geq 2, g(C) \neq 4$. Then $F_X = C - C$. If $g(C) = 4$, then $F_X = C - C \cup \{a - a', a' - a\}$, for a, a' the two g^1_3's on C.*

The genus 2 case is easy, and the genus 3 was proven by Frobenius. I would like to prove the $g(C) = 4$, non hyperelliptic case, but first let's talk a little more about second order theta functions (with zero characteristic). Recall that

$$\tilde{\Theta} = \{\zeta_{g-1} \in \mathrm{Pic}^{g-1}(C) \mid h^0(\zeta_{g-1}) \geq 1\}$$

is a canonical model of the theta divisor of $\mathrm{Jac}(C)$. If $a \in \mathrm{Pic}^d(C)$, then let $\tilde{\Theta}_a = W^0_{g-1} + a \in \mathrm{Pic}^{g-1+d}(C)$. In particular, *the* theta divisor of $\mathrm{Jac}(C)$ is a misnomer, since it's only defined up to translation; but all of its translates are obtainted as $\tilde{\Theta}_{-\zeta}$ where ζ ranges in $\mathrm{Pic}^{g-1}(C)$. For $\zeta \in \mathrm{Pic}^{g-1}(C)$, $\zeta' = K_C - \zeta \in \mathrm{Pic}^{g-1}(C)$, which induces a symmetry (involution) of $\mathrm{Jac}(C)$ (which is isomorphic to $\mathrm{Pic}^{g-1}(C)$). Then $\tilde{\Theta}_{-\zeta'}$ is the image of $\tilde{\Theta}_{-\zeta}$ under the symmetry. Hence, by the Theorem of the Square, divisors of the form $D = \tilde{\Theta}_{-\zeta} + \tilde{\Theta}_{-\zeta'}$ are in $|2\Theta|$. We then have for such D:

$$m_0(D) = m_\zeta(\tilde{\Theta}) + m_{\zeta'}(\tilde{\Theta}) = 2h^0(\mathcal{O}_C(\zeta)).$$

Thus $m_0(D) \geq 4$ if and only if $\zeta \in W^1_{g-1} = Sing(\tilde{\Theta})$. This means that

$$\bigcap_{D\in|2\Theta|,\ m_0(D)\geq 4} D \subseteq \bigcap_{\zeta\in W^1_{g-1}} (\tilde{\Theta}_{-\zeta} + \tilde{\Theta}_{-\zeta'}).$$

Proposition 4. *[3] Let C be a non-hyperelliptic curve of genus 4. Call g^1_3 and h^1_3 its two series of degree 3. Then:*

$$\bigcap_{D\in|2\Theta|,\ m_0(D)\geq 4} D = (C - C) \cup \{\pm(g^1_3 - h^1_3)\}.$$

In particular, if the two series are not coincident, this locus has two isolated points outside the surface $C - C$.

Proof. First note that Frobenius proved that, in general,

$$C - C \subseteq \bigcap_{D\in|2\Theta|,\ m_0(D)\geq 4} D.$$

He showed that divisors of the form $\tilde{\Theta}_{-\zeta} + \tilde{\Theta}_{-\zeta'}$ generate $|2\Theta|$ as a projective space. And the inclusion is easy to check for these kinds of divisors.

By the above comments, the left hand member of the desired equality is contained in

$$\bigcap_{\zeta\in W^1_{g-1}} (\tilde{\Theta}_{-\zeta} + \tilde{\Theta}_{-\zeta'}) = (W^0_3 - g^1_3) \cup (g^1_3 - W^0_3) = (W^0_3 - g^1_3) \cup (W^0_3 - h^1_3).$$

By symmetry, it would then be sufficient to compute its intersection with $W^0_3 - g^1_3$. Let $\mu_3 : C^{(3)} \to \mathrm{Jac}(C) = J$ be defined by $\mu_3(D_3) = D_3 - g^1_3$, then we have a surjection:

$$H^0(J, \mathcal{O}_J(2\Theta)) \to H^0(C^{(3)}, \mu_3^*\mathcal{O}_J(2\Theta)).$$

Write $S \subset C^{(3)}$ as the (set-theoretic) inverse image of $C - C$. Let $d > 0$ be fixed and Λ be a linear system on C of dimension at least $d - 1$. Then we write E_Λ for any divisor of $C^{(d)}$ obtained as $\{D_d \mid D_d \leq \Lambda'\}$, where $\Lambda' \subset \Lambda$ is a subsystem of dimension $d - 1$.

We also get that

$$\mu_3^*\mathcal{O}_J(2\Theta) = \mathcal{O}_{C^{(3)}}(S + E_{|K|}).$$

So the intersection with $W^0_3 - g^1_3$ we are looking for is the image (under μ_3) of $S \cup$ (base locus of $|E_{|K|}|$). The base locus of $|E_{|K|}|$ is $g^1_3 \cup h^1_3 \subset C^{(3)}$, hence we obtain $(C - C) \cup \{h^1_3 - g^1_3\}$. This proves the desired equality.

Now let's show that we actually get two extra, isolated points. Suppose $g^1_3 \neq h^1_3$ and $g^1_3 - h^1_3 \in C - C$. This means that $g^1_3 - h^1_3 \equiv x - y$ for two points $x, y \in C$. Then $g^1_3 + y \equiv h^1_3 + x$. But C has no g^2_4, so this implies that g^1_3 and h^1_3 have members sharing two of their three points. But we saw in class how C is the intersection of a quadric and a cubic, and the two series are the rulings of the quadric. Lines from one ruling only intersect lines from the other ruling in one point. This implies that the two series were coincident, a contradiction. □

References

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