The Jacobi triple product via representation theory

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This is a survey paper presenting the views of Pressley–Segal [1] on the Jacobi triple product formula via representation theory.

1 Introduction

Recall, the *Riemann theta function* is given by the formula

$$\theta(x) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2\tau + 2\pi i n x}$$

and defines an analytic function on $\mathbb{C}$. The Riemann θ–function plays an important role in geometry, algebraic geometry, and number theory—one such role is via sections of holomorphic line bundles over abelian varieties. An alternative formula for θ given by the infinite product expansion

$$\theta(x) = \prod_{n=1}^{n=\infty}(1-q^n)(1+e^{2\pi i x}q^{n-\frac{1}{2}})(1+e^{-2\pi i x}q^{n-\frac{1}{2}}) \qquad (1)$$

where $q = e^{2\pi i\tau}$. The formula (1) is called the *Jacobi triple product formula* and will garner our attention throughout the remainder of this note. Most important for us is the relationship of (1) with 2–dimensional quantum-field theory. Specifically, for a separable Hilbert space $\mathscr{H}$, the formula is obtained from two different views of a representation of a central extension of a subgroup of $\mathrm{GL}(\mathscr{H})$. Indeed, these views correspond to a *fermionic* and *bosonic* construction, and the equivalence of these representations bears (1).

2 Background

2.1 $\mathbf{GL}_{\mathbf{res}}$

Given a separable Hilbert space $\mathscr{H}$, as all such Hilbert spaces are isomorphic, in the sequel we take $\mathscr{H} = \mathrm{L}^2(S^1, \mathbb{C})$. Note that this Hilbert space has a basis given by

$$\{z^n \mid n \in \mathbb{Z},\ z = e^{i\theta}\},$$

and is endowed with a natural decomposition $\mathscr{H} = H_+ \oplus H_-$, where H_+ is spanned by $\{z^n \mid n \geq 0\}$ and H_- is spanned by $\{z^n \mid n < 0\}$. The group of bounded, invertible, linear maps on $\mathscr{H}$ will be denoted by $\mathrm{GL}(\mathscr{H})$. The decomposition of $\mathscr{H}$ affords a decomposition of each $A \in \mathrm{GL}(\mathscr{H})$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Finally, we set $J\colon \mathscr{H} \longrightarrow \mathscr{H}$ to be the map whose restriction to H_+, H_-, respectively, is $\mathrm{Id}, -\mathrm{Id}$.

An operator is *Hilbert–Schmidt* if it satisfies

$$\sum_i \|Ae_i\|^2 < \infty$$

where the $\{e_i\}$ is any orthonormal basis of $\mathscr{H}$, and we define the subgroup $\mathrm{GL}_{\mathrm{res}}(\mathscr{H})$ of $\mathrm{GL}(\mathscr{H})$ to be those operators $A \in \mathrm{GL}(\mathscr{H})$ such that $[A, J]$ is a Hilbert–Schmidt operator. According to the decomposition $\mathscr{H} = H_+ \oplus H_-$, observe that

$$[A, J] = \begin{pmatrix} 0 & -2b \\ 2c & 0 \end{pmatrix}.$$

Consequently, A resides in $\mathrm{GL}_{\mathrm{res}}(\mathscr{H})$ provided b, c are both Hilbert-Schmidt operators. Therefore, $\mathrm{GL}_{\mathrm{res}}(\mathscr{H})$ preserves the decomposition $\mathscr{H} = H_+ \oplus H_-$, up to the image of Hilbert–Schmidt operators.

2.2 A central extension of $\mathbf{GL}_{\mathbf{res}}(\mathscr{H})$

From above, every element of $\mathrm{GL}_{\mathrm{res}}(\mathscr{H})$ is of the form

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where a, d are Fredholm operators and b, c are Hilbert–Schmidt operators. This description of the elements of $\mathrm{GL}_{\mathrm{res}}(\mathscr{H})$ provides a map

$$h\colon \mathrm{GL}_{\mathrm{res}}(\mathscr{H}) \longrightarrow \mathrm{Fred}(H_+)$$

defined by

$$h\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] = a,$$

and we assert h is a homotopy equivalence. To see this, first consider the map

$$h_1 \colon \mathrm{GL}_{\mathrm{res}}(\mathscr{H}) \longrightarrow \mathrm{Fred}(H_+) \times U$$

defined by

$$h_1\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] = \begin{pmatrix} a \\ c \end{pmatrix}.$$

Visibly, this maps onto $\mathrm{Fred}(H_+) \times U$, where U is an open subset of the space $\mathscr{I}_2(H_+, H_-)$ of Hilbert–Schmidt operators from H_+ to H_-. As this space is isomorphic to the homogeneous space $\mathrm{GL}_{\mathrm{res}}(\mathscr{H})/\mathscr{B}$ with

$$\mathscr{B} = \begin{pmatrix} 1 & b \\ 0 & d \end{pmatrix},$$

the map h_1 must be homotopy equivalence. Indeed, the group $\mathscr{B}$ is contractible being a topological product of $\mathrm{GL}(H_+)$ and the vector space of Hilbert–Schmidt operators $\mathscr{I}_2(H_-, H_+)$. In tandem with Kuiper's theorem which provides the contractibility of $\mathrm{GL}(H_+)$, we now see that h_1 is a homotopy equivalence.

It remains to show that h is a homotopy equivalence, a task achieved with the following line of reasoning. The projection

$$\begin{pmatrix} a \\ c \end{pmatrix} \longmapsto a$$

has fiber U_a consisting of those Hilbert–Schmidt operators whose restriction to $\ker(a)$ is one-to-one. As this space is contractible, the projection is a homotopy equivalence, and thus the original map h must be as well.

The components of $\mathrm{GL}_{\mathrm{res}}(\mathscr{H})$ are parameterized by the index of a. Therefore, $\mathrm{GL}_{\mathrm{res}}(\mathscr{H})$ is an extension of the identity component $\mathrm{GL}_{\mathrm{res}}(\mathscr{H})^0$ by $\mathbb{Z}$. To construct a central extension of $\mathrm{GL}_{\mathrm{res}}(\mathscr{H})$, which we denote by $\widetilde{\mathrm{GL}_{\mathrm{res}}(\mathscr{H})}$, we shall first do this over the identity component $\mathrm{GL}_{\mathrm{res}}(\mathscr{H})^0$, and by the $\mathbb{Z}$–action, extend this extension of $\mathrm{GL}_{\mathrm{res}}(\mathscr{H})^0$ to an extension of $\mathrm{GL}_{\mathrm{res}}(\mathscr{H})$. The details are as follows. The identity component of $\mathrm{GL}_{\mathrm{res}}(\mathscr{H})$ is comprised of all of the elements

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

such that $\operatorname{index}(a) = 0$. We also require the subgroup $\mathscr{E}$ of $\mathrm{GL}_{\mathrm{res}}(\mathscr{H}) \times \mathrm{GL}(H_+)$ defined by

$$\mathscr{E} = \left\{ (A, q) \in \mathrm{GL}_{\mathrm{res}}(\mathscr{H}) \times \mathrm{GL}(H_+) \text{ such that } a - q \text{ is of trace class } \right\}.$$

Note, by another application of Kuiper's theorem, we see $\mathscr{E}$ is contractible. With this, we have the extension

$$\mathscr{T} \longrightarrow \mathscr{E} \longrightarrow \mathrm{GL}_{\mathrm{res}}(\mathscr{H})^0$$

over the identity component where $\mathscr{T}$ is the space of operators with a determinant—we say an operator A has *a determinant* if the operator $1 - A$ is of trace class. The elements of $\mathscr{T}$ are invertible operators with determinant, and so there is a homomorphism $\det\colon \mathscr{T} \longrightarrow \mathbb{C}^*$. Setting $\mathscr{T}_1 = \ker(\det)$, this gives a central extension over $\mathrm{GL}_{\mathrm{res}}(\mathscr{H})^0$ which is

$$1 \longrightarrow \mathbb{C}^* \longrightarrow \mathscr{E}/\mathscr{T}_1 \longrightarrow \mathrm{GL}_{\mathrm{res}}(\mathscr{H})^0 \longrightarrow 1.$$

We shall call $\mathscr{E}/\mathscr{T}_1 = \widetilde{\mathrm{GL}_{\mathrm{res}}(\mathscr{H})}^0$, and note with sufficient understanding of the $\mathbb{Z}$–action on $\mathrm{GL}_{\mathrm{res}}(\mathscr{H})$, the production of the desired extension $\widetilde{\mathrm{GL}_{\mathrm{res}}(\mathscr{H})}$ of $\mathrm{GL}_{\mathrm{res}}(\mathscr{H})$ is straightforward.

2.3 Gr($\mathscr{H}$)

To produce the representations of the introduction, we shall require the Grassmannian $\mathrm{Gr}(\mathscr{H})$ of the Hilbert space $\mathscr{H}$ which we now recall. The importance of $\mathrm{Gr}(\mathscr{H})$ here rests with a complex line bundle over $\mathrm{Gr}(\mathscr{H})$ whose dual has an infinite dimensional space of sections which will admit a projective representation of $\mathrm{GL}_{\mathrm{res}}(\mathscr{H})$.

The *Grassmannian* $\mathrm{Gr}(\mathscr{H})$ is the set of all closed subspaces $W \subset \mathscr{H}$ such that the orthogonal projection $\mathrm{pr}_+ \colon W \longrightarrow \mathrm{H}_+$ is a Fredholm operator and the orthogonal projection $\mathrm{pr}_- \colon W \longrightarrow \mathrm{H}_-$ is a Hilbert–Schmidt operator. Alternatively, $\mathrm{Gr}(\mathscr{H})$ is the space of all $W \subset \mathscr{H}$ such that W is in the image of an operator $w\colon \mathrm{H}_+ \longrightarrow \mathscr{H}$ where $\mathrm{pr}_+ \circ w\colon \mathrm{H}_+ \longrightarrow \mathrm{H}_+$ is Fredholm and $\mathrm{pr}_- \circ w\colon \mathrm{H}_+ \longrightarrow \mathrm{H}_-$ is Hilbert–Schmidt.

The following lemma is required in the sequel.

Lemma 2.1. *Let W be an element of $Gr(\mathscr{H})$ and $T\colon W \longrightarrow W^\perp$ a Hilbert–Schmidt operator. Then the graph of T is also an element of $Gr(\mathscr{H})$.*

Proof. Note that the Hilbert–Schmidt operators form an ideal in $\mathrm{GL}(\mathscr{H})$ and the sum of a Fredholm operator and a Hilbert–Schmidt operator is a Fredholm operator. For the latter, notice that Fredholm operators are precisely those operators which are invertible modulo compact operators. As Hilbert–Schmidt operators are compact, the sum of a Fredholm operator and a Hilbert–Schmidt operator remains invertible modulo compact operators.

With this said, we commence with the proof of the lemma. Assume $T\colon W \longrightarrow W^{\perp}$ is a Hilbert–Schmidt operator with $W \in \mathrm{Gr}(\mathscr{H})$. By definition, W is the graph of $w\colon \mathrm{H}_{+} \longrightarrow \mathscr{H}$ with w satisfying the above conditions. Setting $W^{'}$ be the graph of T, the projection $\mathrm{pr}_{+}\colon W^{'} \longrightarrow \mathrm{H}_{+}$ is given by

$$\mathrm{pr}_{+}|W \circ w + \mathrm{pr}_{+}|W \circ T \circ w$$

and is the sum of a Fredholm operator and a Hilbert–Schmidt operator. From what we said above, this implies half of what we require. Namely, pr_{+} is a Fredholm operator. Given that the sum of two Hilbert–Schmidt operators is still a Hilbert–Schmidt operator, a similar argument shows the the other projection $\mathrm{pr}_{-}\colon W^{'} \longrightarrow \mathrm{H}_{-}$ is Hilbert–Schmidt, as required for the lemma. ☐

We now equip $\mathrm{Gr}(\mathscr{H})$ with a smooth manifold structure. To this end, let W be an element of $\mathrm{Gr}(\mathscr{H})$ and U_W be the subset of $\mathrm{Gr}(\mathscr{H})$ consisting of all subspaces that are graphs of a Hilbert–Schmidt operator $W \longrightarrow W^{\perp}$. The set $\{U_W\}_{W\in\mathrm{Gr}(\mathscr{H})}$ is an open cover of $\mathrm{Gr}(\mathscr{H})$ and yield coordinate charts for $\mathrm{Gr}(\mathscr{H})$. For the latter, notice that when $W^{'} \in U_W$ is the graph of $w^{'}$, the map $W^{'} \longrightarrow w'$ is a homeomorphism

$$U_W \longrightarrow \mathscr{I}_2(W, W^{\perp}),$$

where $\mathscr{I}_2(W, W^{\perp})$ is the Hilbert space of Hilbert–Schmidt operators from W to $W^{\perp}$. In total, this yield an especially nice cover of $\mathrm{Gr}(\mathscr{H})$ and will produce the analog of Plücker coordinates in this setting—this hopefully makes an algebraic geometer feel more at home.

Recall that $\mathscr{H} = \mathrm{L}^2(S^1, \mathbb{C})$, and let $\mathscr{S}$ be the set of subsets $S \subset \mathbb{Z}$ such that $\mathrm{card}(S - \mathbb{N}) < \infty$, $\mathrm{card}(\mathbb{N} - S) < \infty$, and H_S be the span of $\{z^s | s \in S\}$.

Lemma 2.2. *Given any $W \in Gr(\mathscr{H})$ there is a $S \in \mathscr{S}$ such that the orthogonal projection $W \longrightarrow H_S$ is an isomorphism.*

Proof. The projection $W \longrightarrow H_{+}$ is a Fredholm operator and hence has finite dimensional kernel. Let $w_1, \ldots, w_n$ span the kernel and choose negative integers $m_1, \ldots, m_n$. Being a Fredholm operator, the co-kernel is finite dimensional, and so we are also given a basis $z^{k_1},\dots,z^{k_r}$ for the orthogonal complement of the image of the projection. With this, define the set

$$S = \mathbb{N} \cup \{m_1,\dots,m_n\} - \{k_1,\dots,k_r\}.$$

Visibly, we can extend the orthogonal projection by mapping $w_{m_i} \longmapsto z^{m_i}$. Consequently, $W \longrightarrow H_S$ is an isomorphism, as asserted. □

We are now ready to introduce Plücker coordinates. For $W \in \mathrm{Gr}(\mathscr{H})$, a sequence $\{w_k\}_{k\geq -d} \in W$ is called an *admissible basis* for W if the linear map $w\colon z^{-d}H_+ \longrightarrow W$ is an isomorphism and the composite $\mathrm{pr}\circ w\colon W \longrightarrow z^{-d}H_+$ where pr is the orthogonal projection, is an operator with determinant. We claim that any W in $\mathrm{Gr}(\mathscr{H})$ possesses an admissible basis. Indeed by the lemma above, there is a set $S \in \mathscr{S}$ such that the orthogonal projection $W \longrightarrow H_S$ is an isomorphism. Upon setting

$$d = \mathrm{card}((S-\mathbb{N}) - \mathrm{card}(\mathbb{N}-S),$$

notice that d is the index of the projection $\mathrm{pr}_+\colon W \longrightarrow H_+$, and so there is an isomorphism that differs from the identity by an operator of finite rank from $z^{-d}H_+ \longrightarrow H_S$. The sought after admissible basis is the image of the z^n under the composition

$$w\colon z^{-d}H_+ \longrightarrow H_S \longrightarrow W.$$

As the composition $\mathrm{pr}\circ w$ differs from the identity by an operator of finite rank, it has a determinant. The composition of two operators with determinant in turn has a determinant. In addition, when $\mathrm{pr}\circ w\colon z^{-d}H_+ \longrightarrow H_+$ has a determinant, so does the map

$$\mathrm{pr}_S \circ w\colon z^{-d}H_+ \longrightarrow H_S,$$

where pr_S is the orthogonal projection onto H_S and $\mathrm{pr}_S\colon W \longrightarrow H_S$ is an isomorphism. Consequently, the operator

$$\mathrm{pr}_S \circ w\colon z^{-d}H_+ \longrightarrow H_S$$

has a determinant since it differs from $\mathrm{pr}\circ w\colon z^{-d}H_+ \longrightarrow H_+$ by an operator of finite rank. With this, we define the *Plücker coordinates* of a point $W \in \mathrm{Gr}(\mathscr{H})$ to be $\pi_S(W) = \det(\mathrm{pr}_S \circ w)$. Setting $\mathscr{V} = \ell^2(\mathscr{S})$, we have the following theorem.

Theorem 2.3. *The Plücker coordinates*$\{\pi_S\}_{S\in\mathscr{S}}$ *give an embedding*

$$\pi\colon Gr(\mathscr{H}) \longrightarrow \mathscr{V}$$

defined by $W \longmapsto \{\pi_S(W)\}_{S\in\mathscr{S}}$.

Proof. A computation shows that when $w\colon z^{-d}H_{+} \longrightarrow W$ defines an admissible basis,

$$\sum_{S\in\mathscr{S}} \|\pi_S(W)\|^2 = \det(w^*w).$$

Thus,

$$\sum_{S\in\mathscr{S}} \|\pi_S(W)\|^2 < \infty$$

and that the map defined by $W \longmapsto \{\pi_S(W)\}_{S\in\mathscr{S}}$ is well defined. For the sake of brevity, the rest of the proof will be omitted. □

Recall, in the finite dimensional case, the components of the Grassmannian are indexed by the dimension of the subspaces representing a point. That approach will not work here since all of the subspaces are infinite dimensional. This is rectified by use of the "virtual dimension. Recall that a point $W \in \mathrm{Gr}(\mathscr{H})$ is the image of a map $w\colon H_{+} \longrightarrow \mathscr{H}$ such that $\mathrm{pr}_{+} \circ w$ is Fredholm. The index of this operator is the virtual index of W. The components of $\mathrm{Gr}(\mathscr{H})$ are indexed by the integers. Two points of $\mathrm{Gr}(\mathscr{H})$ are in the same component if and only if they have the same virtual dimension.

2.4 The determinant line bundle

On a finite dimensional Grassmannian, there is a tautological vector bundle whose fiber over a point is the vector subspace representing that point. We can then take the top exterior power of that vector bundle to get a line bundle, called the *determinant line* over the Grassmannian. The terminology determinant and top exterior power are interchangeable, at least in the finite dimensional case. We will apply this same idea to our infinite dimensional Grassmannian except we will skip the intermediate step of forming the tautological vector bundle and go straight to the determinant line bundle, Det. The determinant line bundle will have no sections but its dual will have an infinite dimensional space of sections. In fact, each of the Plücker coordinates will give a holomorphic section of the dual of the determinant and form a dense subset of Det^*. Admissible bases will play a prominent role.

Construction of Det Let $W \in \mathrm{Gr}(\mathscr{H})$ and w be an admissible basis for W. The fiber of Det at the point W is comprised of formal expressions

$$\lambda w_{-d} \wedge w_{-d+1} \wedge \dots$$

where $\lambda \in \mathbb{C}$. We shall write this as $[\lambda, w]$, and when $w, w^{'}$ are two admissible basis for W, the change of basis transformation t has a determinant and

$$[\lambda, w] = [\lambda \det t, w^{'}].$$

This defines a complex line bundle Det on $\mathrm{Gr}(\mathscr{H})$ and we observe that each of the Plücker coordinates defines a section of Det^* by

$$\pi_S([\lambda, w]) = \lambda \pi_S(W)$$

where w represents an admissible basis for W. Thus Det^* has an infinite dimensional space of sections.

2.5 An action of $\mathbf{GL_{res}}(\mathscr{H})$ on $\mathbf{Gr}(\mathscr{H})$

We assert that there is a transitive action of $\mathrm{GL}_{\mathrm{res}}(\mathscr{H})$ on $\mathrm{Gr}(\mathscr{H})$. For this, if $W \in \mathrm{Gr}(\mathscr{H})$ and $A \in \mathrm{GL}_{\mathrm{res}}(\mathscr{H})$, then $AW \in \mathrm{Gr}(\mathscr{H})$ as $\mathrm{pr}_+|(AW) = \mathrm{pr}_+ \circ a + \mathrm{pr}_+ \circ b$ is a Fredholm operator (since b is a Hilbert–Schmidt operator) and $\mathrm{pr}_-|(AW) = \mathrm{pr}_- \circ c + \mathrm{pr}_- \circ d$ is a Hilbert–Schmidt operator (since the Hilbert–Schmidt operators are a 2–sided ideal in $\mathrm{GL}(\mathscr{H})$). For the transitivity of this action, if $W \in \mathrm{Gr}(\mathscr{H})$, then by definition there exists is a linear map $w\colon H_+ \longrightarrow \mathscr{H}$ with image W. Setting $w^\perp\colon H_- \longrightarrow \mathscr{H}$ with image $W^\perp$, $w_+ = \mathrm{pr}_+ \circ w$, and $w_- = \mathrm{pr}_- \circ w$, the element $A \in \mathrm{GL}_{\mathrm{res}}(\mathscr{H})$ given by

$$A = \begin{pmatrix} w_+ & w_+^\perp \\ w_- & w_-^\perp \end{pmatrix}$$

takes H_+ to W.

As a side note, The stabilizer of H_+ is the subgroup $\mathrm{GL}(H_+) \times \mathrm{GL}(H_-) \subset \mathrm{GL}_{\mathrm{res}}(\mathscr{H})$. Thus,

$$\mathrm{Gr}(\mathscr{H}) \cong \mathrm{GL}_{\mathrm{res}}(\mathscr{H}) / \big(\mathrm{GL}(H_+) \times \mathrm{GL}(H_-)\big)$$

2.6 An action of $\widetilde{\mathbf{GL_{res}}}(\mathscr{H})$ on Det

With this transitive action in hand, it should not be too surprising that this induces an action of $\widetilde{\mathrm{GL}_{\mathrm{res}}}(\mathscr{H})$ on Det. From above, the element of $\widetilde{\mathrm{GL}_{\mathrm{res}}}(\mathscr{H})$ can be represented by pairs $(A, q) \in \mathrm{GL}_{\mathrm{res}}(\mathscr{H}) \times \mathrm{GL}(H_+)$ such that $A - q$ is of trace class. We define the action of $\mathrm{GL}_{\mathrm{res}}(\mathscr{H})$ on Det by

$$(A, q) \cdot [\lambda, w] = [\lambda, Awq^{-1}]$$

where we view w as defining an admissible basis $w\colon z^{-d}H_+ \longrightarrow W$.

3 Representation Theory

Having dispensed with the requisite material, we establish (1) from the introduction.

Previously, we constructed a central extension of $\mathrm{GL}_{\mathrm{res}}(\mathscr{H})$ and produced an action on the complex line bundle Det over $\mathrm{Gr}(\mathscr{H})$. This affords us with an action of $\widetilde{\mathrm{GL}_{\mathrm{res}}(\mathscr{H})}$ on Det^*.

The Plücker coordinates on $\mathrm{Gr}(\mathscr{H})$ produce an infinite dimensional space of sections of Det^*. This, in turn, produces a representation of $\widetilde{\mathrm{GL}_{\mathrm{res}}(\mathscr{H})}$ which will be called the *basic representation* and denote by Γ.

The formula (1) shall now follow from two different constructions of this representation. The first construction, via a fermionic Fock space, arises from a completion of exterior algebras, while the second, via a bosonic Fock space, arises from a completion of symmetric algebras. The details fill the remainder of this note.

3.1 Γ as an exterior algebra

If V is an n–dimensional vector space with Grassmannian

$$\mathrm{Gr}(V) = \bigcup_{k \le n} \mathrm{Gr}_k(V).$$

On each component of $\mathrm{Gr}(V)$, we have the determinant line bundle Det and the space of sections of Det^* is given by $\wedge^k(V^*)$.

Our Hilbert space $\mathscr{H} = L^2(S^1;\mathbb{C})$ has is endowed with a natural filtration

$$\ldots H_{-2} \subset H_{-1} \subset H_0 \subset H_1 \subset H_2 \ldots$$

where

$$H_n \subset z^n H_+.$$

Setting $H_{-n,n} = H_n/H_{-n}$, subspaces of H sandwiched between H_{-n} and H_n correspond to subspaces of $H_{-n,n}$ and the finite dimensional Grassmannians $\mathrm{Gr}(H_{-n,n})$ correspond to subspaces of $\mathrm{Gr}(\mathscr{H})$. We will take the following proposition as a fact.

Proposition 3.1. *The $Gr(H_{-n,n})$'s are dense in $Gr(\mathscr{H})$.*

For each n, restriction gives us a map $\Gamma \longrightarrow \Lambda(H_{-n,n})$. This holds for every n and so we get a map $\Gamma \longrightarrow \lim_{\leftarrow} \Lambda(H^*_{-n,n})$ which is injective since the set of $\Lambda(H_{-n,n})$ are dense. As $H_{-n,n} \cong H_{-n,0} \oplus H_{0,n}$, we can identify $\Lambda(H_{0,n}) \cong \Lambda(H^*_{0,n})$. Thus, $\Lambda(H_{-n,n}) \cong \Lambda(H^*_{-n,0}) \otimes \Lambda(H_{0,n})$, and taking limits we get an identification

$$\lim_{\leftarrow} \Lambda(H^*_{-n,n}) \subset \Lambda(H^*_{-}) \otimes \Lambda(H_+)$$

The space $\mathscr{H} = L^2(S^1;\mathbb{C})$ possesses a natural S^1 action and hence $\mathscr{H}$ decomposes into eigenspaces for the S^1–action. The elements of S^1 which act by $e^{-ik\theta}$ are called the *elements of energy k*. We now see that $\Lambda(H_{0,n})$ and $\Lambda(H^*_{-n,0})$ consist of elements of positive energy $\leq n$, and so Γ is a dense subset of $\lim_{\leftarrow} \Lambda(H^*_{-n,n})$

3.2 Γ as a symmetric algebra

The Hilbert space $\mathscr{H} = L^2(S^1;\mathbb{C})$ also admits an action by the loop group $L\mathbb{C}^*$ given by multiplication. We let N^- be the subgroup of elements of the form

$$1 + h_1 z^{-1} + h_2 z^{-2} + \dots$$

with $h_i \in \mathbb{C}$. The orbit of H_+ under this action does not meet H_- which is fixed under this action. Thus all of the point W in the orbit of N^- project isomorphically onto H_+. In particular, we can trivialize the bundle Det restricted to the orbit of N_- by identifying $\mathrm{Det}(W)$ with $\mathrm{Det}(H_+)$. Thus holomorphic sections restricted to this orbit can be identified with holomorphic functions on N^- and we get a map $\Gamma_0 \longrightarrow \mathrm{Hol}(N^-)$, where Γ_0 refers to the sections restricted to the component of $\mathrm{Gr}(\mathscr{H})$ consisting of points of virtual dimension 0. The polynomial ring $\mathbb{C}[h_1, h_2, \dots]$ forms a dense subset of $\mathrm{Hol}(N^-)$ and with notation, we have the following proposition.

Proposition 3.2. *The Plücker coordinates* $\{\pi_S\}_{S\in\mathscr{S}}$ *form a basis for* $\mathbb{C}[h_1, h_2, \dots]$.

As the Plücker coordinates form an dense subset of Γ, the image $\Gamma_0 \longrightarrow \mathrm{Hol}(N^-)$ forms a dense subset and induces an isomorphism on the elements of finite energy. For the Lie algebra A of N^-, the exponential map is an isomorphism and permits us the identification $\mathrm{Hol}(A) \cong \mathrm{Hol}(N^-)$. Since the symmetric algebra $S(A)$ sits inside of $\mathrm{Hol}(A)$ as a dense subset, we identify Γ with a completed symmetric algebra $\hat{S}(A)$. All of this was done over the component of index 0, and so more specifically, we have described Γ_0 as a completed symmetric algebra $\hat{S}(A)_0$. As before, we get the full representation by translating to the other components.

3.3 The Jacobi Triple Product

The isomorphism of Hilbert spaces

$$\bigoplus_{d\in\mathbb{Z}} \hat{S}(A)_{(d)} \cong \hat{\Lambda}(H_+ \oplus H_-)$$

is an isomorphism of $\widetilde{\mathrm{GL}_{\mathrm{res}}(\mathscr{H})}$–spaces, since both were constructed as models of Γ. The group $\widetilde{\mathrm{GL}_{\mathrm{res}}(\mathscr{H})}$ contains a subgroup which is the loop group $L\mathbb{T}$ and act by multiplication. Of course, $\mathscr{H}$ admits an action by $\mathbb{T}$ since $\mathscr{H}$ is $L^2(S^1)$. These actions combine to give an action of $\mathbb{T}\times\mathbb{T}$ on both of Hilbert spaces. Since $\mathbb{T}\times\mathbb{T}$ is a compact abelian group, both representations break up into a sum of one-dimensional representations of $\mathbb{T}\times\mathbb{T}$, say $\bigoplus_{k\in\mathbb{Z}} H_k$, and form a dense subset of the Hilbert space. The k^{th} eigenspace of $\mathbb{T}\times\mathbb{T}$ has the character $u\cdot z^k$, where u comes form the loop group and z is $\mathbb{T}$ acting on S^1 by rotation.[1]

Proposition 3.3. *If a vector space V is a sum of one-dimensional representations with character ρ_i, then the representation $\Lambda(V)$ has character*

$$\prod_i (1-\rho_i)$$

while $S(V)$ has character

$$\prod_i (1-\rho_i)^{-1}.$$

We apply the above proposition to $\bigoplus_{d\in\mathbb{Z}} \hat{S}(A)_{(d)}$ and $\hat{\Lambda}(H_+\oplus H_-)$. For the latter, by Proposition 3.3, $\hat{\Lambda}(H_+\oplus H_-)$ has character

$$\prod_{k\geq 0}(1+uz^k)\prod_{k>0}(1+u^{-1}z^k).$$

For the former, by Proposition 3.3, $\hat{S}(A)_{(0)}$ has character

$$\prod_{k\geq 0}(1-z^k)^{-1}$$

and so $\hat{S}(A)_{(d)}$ has the character

$$u^{-d}z^{\frac{1}{2}d(d+1)}\prod_{k\geq 0}(1-z^k)^{-1}.$$

Therefore $\bigoplus_{d\in\mathbb{Z}} \hat{S}(A)_{(d)}$ has the character

$$\sum_{d\in\mathbb{Z}} u^{-d}z^{\frac{1}{2}d(d+1)}\prod_{k\geq 0}(1-z^k)^{-1}.$$

Equating the two expressions and rearranging we get

$$\sum_{d\in\mathbb{Z}} u^{-d}z^{\frac{1}{2}d(d+1)} = \prod_{k>0}\left\{(1+uz^k)(1-z^k)(1+u^{-1}z^k)\right\}.$$

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[1] This is the group action which breaks up $L^2(S^1)$ into the familiar one dimensional eigenspaces.

Our original expression for the theta function is

$$\theta(x) = \sum_{n \in \mathbb{Z}} e^{\pi i n^2 \tau + 2\pi i n x}$$

with the product formulation

$$\theta(x) = \prod_{n=1}^{n=\infty} (1 - q^n)(1 + e^{2\pi i x} q^{n-\frac{1}{2}})(1 + e^{-2\pi i x} q^{n-\frac{1}{2}}).$$

This expression upon setting $u = e^{2\pi i x} \cdot e^{-\pi i \tau}$ and $z = e^{2\pi i \tau}$ recovers (1).

References

[1] Andrew Pressley and Graeme Segal, *Loop groups*, Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, 1986, Oxford Science Publications.

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