PRYM VARIETIES

PETAR IVANOV AND PAUL LARSEN

Contents

1. Introduction

Prym varieties arise in the context of double covers of curves. If $\pi : \widetilde{C} \to C$ is a double cover, the Prym variety of the covering is a (principally polarized) subvariety of $Pic^0(\widetilde{C}) = \widetilde{J}$. The problem of characterizing Prym varieties among all principally polarized abelian varieties is a classical one, originating from the Riemann-Schottky problem. More recently, Prym varieties have been studied in the context of mathematical physics, for example, in relation to Calabi-Yau three-folds and string theory ([GHL]) as well as to dynamical systems ([Mc]).

Prym varieties belong to a larger class known as Prym-Tyurin varieties. A Prym-Tyurin variety for a curve $\widetilde{C}$ is principally polarized abelian subvariety (Z, Ξ) of $(\widetilde{J}, \widetilde{\Theta})$, such that the restriction of $\widetilde{\Theta}$ to Z is a multiple of Ξ. For a finite morphism of curves, $\pi : \widetilde{C} \to C$, if the complementary subvariety to $Im\pi^*$ is a Prym-Tyurin variety, it is called a Prym variety. A theorem of Mumford says that there are exactly three morphisms that lead to Prym varieties: étale double coverings, double coverings with two branch points and genus 2 coverings of an elliptic curve. We will mainly consider the first case of étale coverings. We also provide an explicit topological construction of a Prym-Tyurin associated to an étale double cover, which gives a geomtrical intuition.

Throughout we assume that the basefield is $\mathbb{C}$, though most of the results hold for any field of characteristic other than 2. We also assume that the genus of the base curve C is at least one to avoid trivialities. One final comment on notation: objects defined on the covering curve $\widetilde{C}$ will be denoted with a tilde, so $\widetilde{J} := Pic^0(\widetilde{C}) = Jac(\widetilde{C})$, $\tilde{g} := genus(\widetilde{C})$, etc..

2. Complementary Abelian Subvarieties

Let (X, L) be a polarized abelian variety X (we will be dealing with Jacobians, so the polarization will be principal). Let $i : Y \hookrightarrow X$ be an abelian subvariety. Recall that there is an isogeny $\phi_L : X \to \hat{X}$ and that the exponent of X, $e(X)$, is the exponent of $\ker \phi_L$. Define $e(Y) := e(i^*L)$. Recall that the inverse isogeny is given by $\psi_{i^*L} = e(Y)\phi_{i^*L}^{-1}$.

Definition 2.1. *Define the* norm-endomorphism *of* X *associated to* Y *(with respect to* L*) as the map* $N_Y \in End(X)$ *defined by*

$$N_Y : X \xrightarrow{\phi_L} \hat{X} \xrightarrow{\hat{i}} \hat{Y} \xrightarrow{\psi_{i^*L}} Y \xrightarrow{i} X , \tag{2.1}$$

i.e., $N_Y = i \circ \psi_{i^*L} \circ \hat{i} \circ \phi_L$.

Lemma 2.2. $N'_Y = N_Y$ *and* $N_Y^2 = e(Y)N_Y$*, where* $'$ *denotes the Rosati involution with respect to* L.

Proof. By definition $N'_Y = \phi_L^{-1} \circ \hat{N}_Y \circ \phi_L$. Since $\hat{}$ is a contravariant functor, $\hat{N}_Y = \hat{\phi}_L \circ i \circ \hat{\psi}_{i^*L} \circ \hat{i}$. We know that $\hat{\phi}_L = \phi_L$ under the canonical identification $\hat{\hat{X}} = X$ and also $\hat{\psi}_{i^*L} = \psi_{i^*L}$. Thus $N'_Y = i \circ \psi_{i^*L} \circ \hat{i} \circ \phi_L = N_Y$.

For the second equality, since $\hat{i} \circ \phi_L i = \phi_{i^*L}$,

$$\begin{aligned} N_Y^2 &= i \circ \psi_{i^*L} \circ \hat{i} \circ \phi_L \circ i \circ \psi_{i^*L} \circ \hat{i} \circ \phi_L \\ &= e(Y) i \circ \phi_{i^*L}^{-1} \circ \phi_{i^*L} \circ \psi_{i^*L} \hat{i} \circ \phi_L \\ &= e(Y) i \circ \psi_{i^*L} \circ \hat{i} \circ \phi_L \\ &= e(Y) N_Y, \end{aligned} \tag{2.2}$$

as desired. ◻

Thus N_Y is symmetric (with respect to Rosati involution) and almost idempotent. Extending to the rationals we can define true symmetric idempotent with respect to Y, $\varepsilon_Y \in End_{\mathbb{Q}}(X)$, by

$$\varepsilon_Y := \frac{1}{e(Y)} N_Y = i \circ \phi_{i^*L}^{-1} \circ \hat{i} \circ \phi_L. \tag{2.3}$$

Conversely, if ε is any symmetric idempotent in $End_{\mathbb{Q}}(X)$, then there exists $n > 0$, such that $n\varepsilon \in End(X)$. Then define the subvariety of X: $X^\varepsilon := Im(n\varepsilon)$.

Lemma 2.3. X^ε *does not depend on the choice of* n*, and the maps* $Y \mapsto \varepsilon_Y$ *and* $\varepsilon \mapsto X^\varepsilon$ *are inverse to each other.*

Definition 2.4. *The* complementary abelian variety Z *to* Y *in* X *is defined as* $Z := X^{1-\varepsilon_Y}$.

Because of Lemma 2.3, $\varepsilon_Z = \varepsilon_{X^{1-\varepsilon_Y}} = 1 - \varepsilon_Y$. Thus $X^{1-\varepsilon_Z} = X^{\varepsilon_Y} = Y$, so Y is also complementary to Z.

Using the definitions and the lemma, we can easily verify the following properties of the norm-endomorphisms:

Lemma 2.5. (1) $N_Y|_Y = e(Y) id_Y$,
(2) $N_Y|_Z = 0$,
(3) $N_Y N_Z = 0$, *and*
(4) $e(Y)N_Z + e(Z)N_Y = e(Y)e(Z) id_X$.

The last relation is the most important. In the case when X is principally polarized, one can prove that $e(Y) = e(Z) =: e$. Then $N_Z + N_Y = e\ id_X$. In addition, the Poincaré reducibility theorem (stating that $(N_Y, N_Z) : X \to Y \times Z$ is an isogeny) is a consequence of it. In particular $\dim X = \dim Y + \dim Z$.

The constructions from this section now will be applied in the special case of $(X, L) = (\widetilde{J}, \widetilde{\Theta})$.

3. Definition of the Prym Variety

Let $\pi : \widetilde{C} \to C$ be a double covering of curves, which is étale or ramified in two points, and $\pi^* : J = \mathrm{Pic}^0(C) \to \widetilde{J} = \mathrm{Pic}^0(\widetilde{C})$ the pullback map on their Jacobians. On the level of divisors, π^* acts as $\pi^{-1} : np \mapsto n\tilde{p} + n\tilde{p}'$ if $\tilde{p}$ and $\tilde{p}'$ are the two distinct points that lie above p and $np \mapsto 2n\tilde{p}$ if $\tilde{p}$ is a ramification point of degree 2 above p. Let the image be $Y := \pi^*J$ and Z be the complementary subvariety to Y in $\widetilde{J}$. The exponents of Y and Z are both equal to 2. Thus by the note made at the end of the last section, $N_Y + N_Z = 2id_{\widetilde{J}}$.

The norm map $Nm : \widetilde{J} \to J$ associated to the covering is defined as follows: $Nm([n\tilde{p}]) = [n\pi(\tilde{p})] = [np]$. One can easily check that

$$Nm \circ \pi^* = [2]_J. \tag{3.1}$$

We now consider the kernel of π^*. It is easy to see that $\ker \pi^* \subset J_2 = \{\text{points of order 2 in } J\}$. Indeed, $\pi^*([D]) = 0$ implies $2[D] = Nm(\pi^*([D])) = 0$. It turns out that if π is not étale $\ker \pi^* = 0$ and if π is étale there is only one nontrivial element in that kernel, which means that given the covering π, there is uniquely defined element $\eta \in J_2$. In particular, π^* is an isogeny form J to its image Y in $\widetilde{J}$. Conversely, any $\eta \in J_2$ defines an étale double covering $\pi_\eta : C_\eta \to C$. ([Hart], IV, Ex. 2.7).

Lemma 3.1. *The norm map and the norm-endomorphism N_Y (with respect to the principal polarization $\widetilde{\Theta}$ on $\widetilde{J}$) are related as follows: $N_Y = \pi^* \circ Nm$.*

Proof. Let j be the isogeny and $i : Y \hookrightarrow \widetilde{J}$ the inclusion, such that $\pi^* = i \circ j$. If now we apply the functor ˆ we get the following commutative diagram.

$$\begin{array}{ccccc} J & \xrightarrow{j} & Y & \xrightarrow{i} & \widetilde{J} \\ {\scriptstyle [2]_J}\downarrow & & \downarrow{\scriptstyle \phi_{i^*\widetilde{\Theta}}} & & \downarrow{\scriptstyle \phi_{\widetilde{\Theta}}\ \simeq} \\ J & \xleftarrow[\hat{j}]{} & \hat{Y} & \xleftarrow[\hat{i}]{} & \hat{\widetilde{J}} \end{array}$$

Note that the right vertical arrow is isomorphism, because $\widetilde{\Theta}$ is principal polarization and the left vertical arrow is multiplication by 2, because the pullback of $\widetilde{\Theta}$ through π^* is twice the principal polarization on J, i.e. $\phi_{(\pi^*)^*\widetilde{\Theta}} = \phi_{2\Theta} = 2\phi_\Theta$ and we identify J and $\hat{J}$ through ϕ_Θ. Moreover, if we follow the arrows from $\widetilde{J}$ to J in the diagram above we get the norm map Nm. If j^{-1}, $\hat{j}^{-1}$ denote the inverses of j, $\hat{j}$ in $Hom_{\mathbb{Q}}(Y, J)$, $Hom_{\mathbb{Q}}(J, \hat{Y})$, then from the left square in the diagram we get $\phi_{i^*\widetilde{\Theta}} = 2\hat{j}^{-1} \circ j^{-1}$. Now $\psi_{i^*\widetilde{\Theta}} = 2\phi^{-1}_{i^*\widetilde{\Theta}}$, so $\psi_{i^*\widetilde{\Theta}} = j \circ \hat{j}$. Finally, following the diagram we get $\pi^*Nm = i \circ j \circ \hat{j} \circ \hat{i} \circ \phi_{\widetilde{\Theta}} = i \circ \psi_{i^*\widetilde{\Theta}} \circ \hat{i} \circ \phi_{\widetilde{\Theta}} = N_Y$. □

Consider the involution $\iota : \widetilde{C} \to \widetilde{C}$ that interchanges the preimages of any point $p \in C$ (it doesn't move the 2-ramification points on $\widetilde{C}$). This map extends by linearity to divisors on $\widetilde{C}$, and hence to an involution $\iota : \widetilde{J} \to \widetilde{J}$.

In terms of divisor classes on $\widetilde{C}$ (i.e., in terms of points in $\widetilde{J}$), the involution can be written

$$\iota([\widetilde{D}]) = \pi^*(Nm([\widetilde{D}]) - [\widetilde{D}], \tag{3.2}$$

where $[\tilde{D}] = \{\tilde{D} + (f) | f \in \mathcal{M}(\widetilde{C})\}$ denote the divisor class of D. Indeed, if $\widetilde{D} = n\tilde{p}$, where $\tilde{p}$ is unramified point on $\widetilde{C}$,then $\pi^*(Nm(\widetilde{D})) - \widetilde{D} = \pi^*(np) - n\tilde{p} = n\tilde{p} + n\tilde{p}' - n\tilde{p} = n\tilde{p}'$, and if $\widetilde{D} = n\tilde{p}$, where $\tilde{p}$ is ramification point, then $\pi^*(Nm(\widetilde{D})) - \widetilde{D} = \pi^*(np) - n\tilde{p} = 2n\tilde{p} - n\tilde{p} = n\tilde{p}$.

In particular, by Lemma 3.1, $N_Y = 1 + \iota$ and hence $N_Z = 2 - N_Y = 1 - \iota$. Thus $Y = ImN_Y = Im(1 + \iota)$ and $Z = ImN_Z = Im(1 - \iota)$.

Note that if $\widetilde{D}_+ = \sum n_i(\tilde{p}_i + \iota\tilde{p}_i) \in Div(\widetilde{C})$, then $\iota\widetilde{D}_+ = \widetilde{D}_+$, in other words $[\widetilde{D}_+]$ is in $\widetilde{J}_+$ - the $(+1)$-eigenspace of ι. For $\widetilde{D}_- = \sum n_i(\tilde{p}_i - \iota\tilde{p}_i) \in Div(\widetilde{C})$, however, $\iota\widetilde{D}_- = -\widetilde{D}_-$, so $[\widetilde{D}_-]$ is in $\widetilde{J}_-$ - the (-1)-eigenspace of ι.

Proposition 3.2. *$\iota([\widetilde{D}]) = [\widetilde{D}]$ for $[\widetilde{D}] \in \pi^*J = Y$ and $\iota([\widetilde{D}]) = -[\widetilde{D}]$ if $[\widetilde{D}] \in \ker Nm$, in other words $Y \subset \widetilde{J}_+$ and $\ker Nm \subset \widetilde{J}_-$.*

Proof. The first statement has already been proved once we note that line bundles associated to divisors of the form $\widetilde{D}_+$ are precisely the elements of π^*J. Also, shortly, $\iota \circ (1 + \iota) = \iota + \iota^2 = \iota + 1$, so ι fixes the points in $Im(1 + \iota) = Y$. The second follows directly from equation (3.2), because if $\widetilde{D} \in \ker Nm$, then $\iota([\widetilde{D}]) = [-\widetilde{D}] = -[\widetilde{D}]$. □

Definition 3.3. *The* Prym variety *of $\widetilde{C}$ over C is defined*

$$P = (\ker Nm)^0. \tag{3.3}$$

Lemma 3.4. *$Im(1 - \iota) = Z = \ker(1 + \iota)^0 = (\widetilde{J}_-)^0 = (\ker Nm)^0 = P$.*

Proof. Recall that $N_Y = 1 + \iota$ and $N_Z = 1 - \iota$. The first, the third and the fifth equalities are definitions. Also $\dim\ker N_Y = \dim\widetilde{J} - \dim ImN_Y = \dim\widetilde{J} - \dim Y = \dim Z = \dim\widetilde{J} - \dim J$, because $\dim J = \dim Y$ since π^* is isogeny. Now the second equation follows from Lemma 2.5: $N_Y N_Z = 0$ and hence $Z = ImN_Z \subset (\ker N_Y)^0$, because ImN_Z is connected and contain 0. But the inclusion is actually equality, because the dimensions are the same. From Lemma 3.1 follows that $\ker Nm \subset \ker N_Y$ and hence $(\ker Nm)^0 \subset (\ker N_Y)^0 = Z$. Nm is onto J, so $\dim\ker Nm = \dim\widetilde{J} - \dim J = \dim Z$ and again the fourth equality is true because of the equal dimensions. □

Proposition 3.5. *$\dim P = g + n - 1$, where g is the genus of C and $n' = 2n$ is the number of branch points.*

Proof. Each branch point has multiplicity 2, so by Hurwitz's formula

$$2 - 2\tilde{g} = \chi(\widetilde{C}) = 2\chi(C) - \#\text{ branch points} = 4 - 4g - 2n.$$

Hence the genus of $\widetilde{C}$ is $\tilde{g} = 2g + n - 1$ (This argument also shows that the number of branch points must be even). Thus $\dim J = g$ and $\dim\widetilde{J} = 2g + n - 1$. Now $\dim P = \dim\ker Nm = \dim\widetilde{J} - \dim J = g + n - 1$, since Nm is obviously surjective. □

4. The topological construction for étale coverings

Now let π be an étale covering. Let C be a curve of genus $g+1$, $g \geq 0$; the covering is determined by a nonzero element $\mu_0 \in H_1(C, \mathbb{Z}_2)$. In other words C, is cut along μ_0 and then two copies are glued together with upper and lower boundary reversed, such that the orientations fit. The resulting curve $\widetilde{C}$ has genus $2g+1$. We can choose a symplectic basis $\lambda_0, \ldots, \lambda_g, \mu_0, \ldots, \mu_g$ for $H_1(C, \mathbb{Z})$ (the image of μ_0 in $H_1(C, \mathbb{Z}_2)$ is the cycle that we cut along), i.e. such that $I(\lambda_i, \mu_j) = -I(\mu_j, \lambda_i) = \delta_{ij}$. Recall that the Jacobian is given by $\widetilde{J} = H^0(\widetilde{C}, \Omega^1_{\widetilde{C}})^\vee / H_1(\widetilde{C}, \mathbb{Z})$, where the embedding of the lattice $H_1(\widetilde{C}, \mathbb{Z})$ is induced from the map $\gamma \mapsto (\omega \mapsto \int_\gamma \omega)$. Then the Prym variety is given by $P = H^0(\widetilde{C}, \Omega^1_{\widetilde{C}})^\vee_- / H_1(\widetilde{C}, \mathbb{Z})_-$, where by lower index $-$ we denote the (-1)-eigenspaces of the corresponding involutions. In this context the involution ι is the map interchanging the two copies, so for $1 \leq i \leq g$ we have the cycles $\lambda_i^+, \lambda_i^- = \iota\lambda_i^+$ and $\mu_i^+, \mu_i^- = \iota\mu_i^+$ in $H_1(\widetilde{C}, \mathbb{Z})$, which together with $\tilde{\lambda_0} = \iota\tilde{\lambda_0}$ and $\tilde{\mu_0} = \iota\tilde{\mu_0}$ form a symplectic basis for $H_1(\widetilde{C}, \mathbb{Z})$. We have $\pi_*(\lambda_i^\pm) = \lambda_i$, $\pi_*(\mu_i^\pm) = \mu_i$, while $\pi_*(\tilde{\lambda_0}) = 2\lambda_0$, $\pi_*(\tilde{\mu_0}) = \mu_0$.

Now it is clear that $\alpha_i := \lambda_i^+ - \lambda_i^-$, $\beta_i := \mu_i^+ - \mu_i^-$, $i = 1, \ldots, g$ is a basis for $H_1(\widetilde{C}, \mathbb{Z})_-$. Indeed, these vectors together with the $(+1)$-eigenvectors $\lambda_i^+ + \lambda_i^-$, $\mu_i^+ + \mu_i^-$, $\tilde{\lambda}_0$ and $\tilde{\mu}_0$ are $4g+2$ linearly independent vectors, i.e., a basis for $H_1(\widetilde{C}, \mathbb{Z})$. Thus, in particular, the dimension of P is $g = g(C) - 1$. Now, having the basis explicitly we can calculate E in this basis, where E is the alternating 2-form associated to $\widetilde{J}$:

$$E(\alpha_i, \beta_j) = 2\delta_{ij}, E(\alpha_i, \alpha_j) = E(\beta_i, \beta_j) = 0.$$

Thus the induced polarization is two times a principal polarization, hence P is Prym-Tyurin variety.

Note that this is consistent with the formula that we derived in the previous section, because here the genus of C is $g+1$ and $n = 0$.

5. Singularities of the theta divisor of P

Let $J_n = \mathrm{Pic^n(C)}$ and $\widetilde{J}_n = \mathrm{Pic^n(\widetilde{C})}$ be the degree n line bundles on C and $\widetilde{C}$. After picking a divisor class (which we identify with the corresponding line bundle) $L_n \in J_n$ of degree n, we have an isomorphism between J and J_n, given by the translation in Pic(C): $L \mapsto L \otimes L_n^{-1}$ (and similarly for $\widetilde{J}$ and $\widetilde{J}_n$). In particular, J_n and $\widetilde{J}_n$ are principal homogeneous spaces for J and $\widetilde{J}$, respectively.

It turns out that it is convenient to take $n = 2g-2$, the chosen elements to be the canonical divisors on J and $\widetilde{J}$, namely K_C and $K_{\widetilde{C}}$, and the norm map to be seen as $Nm : \widetilde{J}_{2g-2} \to J_{2g-2}$. In other words, if $L \in \widetilde{J}_{2g-2}$, then

$$Nm(L) = Nm(L \otimes K_{\widetilde{C}}^\vee) \otimes K_C.$$

Proposition 5.1. *$L \in \ker Nm$ if and only if $L \otimes K_{\widetilde{C}} \in Nm^{-1}(K_C)$.*

Proof. Let $L \in \widetilde{J}$ and suppose $Nm(L) = \mathcal{O}_C$. Then $Nm(L \otimes K_{\widetilde{C}}) = Nm(L) \otimes K_C = K_C$. For the other direction, if $L' = L \otimes K_{\widetilde{C}}$ and $Nm(L') = K_C$, then $Nm(L) = Nm(L' \otimes K_{\widetilde{C}}^\vee) = Nm(L' \otimes K_{\widetilde{C}}^\vee) \otimes K_C \otimes K_C^\vee = Nm(L') \otimes K_C^\vee = K_C \otimes K_C^\vee = \mathcal{O}_C$. □

Thus $\ker Nm$ corresponds to $Nm^{-1}(K_C)$ via a translation in $\mathrm{Pic}(\widetilde{\mathrm{C}})$. In particular, by abusing the notation we can write $P \subset Nm^{-1}(K_C)$. The following result ([M1]) explains why we consider the principal homogeneous spaces.

Theorem 5.2. *$Nm^{-1}(K_C) \subset \widetilde{J}$ breaks into two components P^+ and P^-, such that for $L_n \in Nm^{-1}(K_C)$, $h^0(\widetilde{C}, L_n)$ is even/odd if and only if $L_n \in P^+/P^-$. Moreover, $P^+ \subset \widetilde{J}_{2g-2}$ corresponds to $P \subset \widetilde{J}$.*

On the other hand, by Riemann's theorems, the theta divisor on $\widetilde{J}$ is a translation by some theta characteristic $\kappa \in \widetilde{J}_{\tilde{g}-1}$ of the image of $\widetilde{C}_{g-1}$ under the Abel-Jacobi map, i.e., $\widetilde{\Theta} = W_{\tilde{g}-1} - \kappa$. Then $mult_{L_n}(W_{\tilde{g}-1}) = h^0(\widetilde{C}, L_n)$ and any point $L \in \widetilde{J}$, which is in P has even multiplicity in $\widetilde{\Theta} = W_{\tilde{g}-1} - \kappa$ (multiplicity 0 means $L \notin \widetilde{\Theta}$).

The goal of the remainder of this paper is to characterize the singularities of the theta divisor of the Prym variety P in terms of the theta divisor of the Jacobian containing P. The main tool will be Theorem 5.2, or rather a slight reformulation of it, Theorem 5.5, below. Let $i_P : P \hookrightarrow \widetilde{J}$ be the inclusion map. The precise statement of the desired result is as follows:

Proposition 5.3. *If Ξ is a theta divisor defining the principal polarization of a Prym variety P such that $2\Xi = i_P^*\widetilde{\Theta}$, then*

$$sing\ \Xi = \{x \in P | mult_x\widetilde{\Theta} \geq 4\} \cup \{x \in P | mult_x\widetilde{\Theta} = 2, T_xP \subseteq TC_x\widetilde{\Theta}\}, \tag{5.1}$$

where $TC_x\widetilde{\Theta}$ is the tangent cone at x.

$\ker Nm$ consists of two components, one of which is the Prym variety P. Denote the other by P_1. In the sense of Theorem 5.2, these are P^+ and P^- up to translation.

Recall that a theta characteristic is a line bundle $\widetilde{L} \in \mathrm{Pic}(\widetilde{\mathrm{C}})$ such that $\widetilde{L} \otimes \widetilde{L} = K_{\widetilde{C}}$. In particular, $deg(\widetilde{L}) = \tilde{g}-1$, where $\tilde{g}$ is the genus of $\widetilde{C}$, since $deg(K_{\widetilde{C}}) = 2\tilde{g}-2$. We say that a theta characteristic L is *even* if its space of sections $H^0(\widetilde{C}, \widetilde{L})$ has even dimension. If $h^0(\widetilde{C}, \widetilde{L}) = \dim H^0(\widetilde{C}, \widetilde{L})$ is odd, then we say that $\widetilde{L}$ is *odd*.

We merely state the first lemma that builds toward a proof of Proposition 5.3, since it requires results about symmetric divisors and theta characteristics that take us too far afield. See [BL], sections 4.7, 11.2.1 and 12.6 for background and proofs.

Lemma 5.4. *There exists an even theta characteristic $\tilde{\kappa}$ on $\widetilde{C}$ that is the pull-back of a theta characteristic κ on C.*

The following theorem–which characterizes the Prym variety and its non-trivial coset in terms of even and odd theta characteristics–will be the main step toward proving the desired characterization of Ξ. Using the lemma, fix $\tilde{\kappa}$, an even theta characteristic on $\widetilde{C}$ that is the pull-back of theta characteristic on C.

Theorem 5.5. $P = \{\widetilde{L} \in \ker Nm | h^0(\widetilde{L} \otimes \tilde{\kappa}) \equiv 0(2)\}$ *, and*
$P_1 = \{\widetilde{L} \in \ker Nm | h^0(\widetilde{L} \otimes \tilde{\kappa}) \equiv 1(2)\}$.

Proof. We will prove the theorem in four stages.
Step 1. $h^0(\widetilde{L} \otimes \tilde{\kappa})$ is constant mod (2) on P and P_1.

Proof. We do not offer a self-contained proof here, but rather appeal to a theorem of Mumford regarding quadratic forms on vector bundles over curves (see [M1], [Har] and [ACGH] for a discussion). This theorem will enable us to construct a quadratic form on the vector bundle $\pi_*\widetilde{L}$ over C, where the quadratic form takes values i n the canonical bundle K_C. This quadratic form will be constant on P and P_1, which will give the desired result.

Since $\widetilde{L} \in \ker Nm$, $\mathcal{O}_{\widetilde{C}} = \pi^* Nm(\widetilde{L}) = \widetilde{L} \otimes \iota^* \widetilde{L}$. Hence if $U \subseteq C$ is an open set, and $\sigma, \sigma' \in H^0(U, \pi_*\widetilde{L}) = H^0(\pi^{-1}(U), \widetilde{L})$, it follows that

$$\sigma \otimes \iota^* \sigma' \in H^0(\pi^{-1}(U), \mathcal{O}_{\widetilde{C}}). \tag{5.2}$$

Abusing notation, we denote the usual norm map on extensions of function fields also by Nm, that is, for f in the function field of $\widetilde{C}$, $(Nm\ f)(z) = \Pi_{\tilde{z} \in \pi^{-1}(z)} f(\tilde{z})$. We define $B(\sigma, \sigma') = Nm(\sigma \otimes \iota^* \sigma')$: this is a section of $\mathcal{O}_C$ over U, and defines a non-degenerate bilinear form on $\pi_*\widetilde{L}$ with values in $\mathcal{O}_c$. Pre-composing with the diagonal map, we get a non-degenerate quadratic form on $\pi_*\widetilde{L}$. If we tensor $\pi_*\widetilde{L}$ with κ, then B defines a quadratic form on $\pi_*\widetilde{L}$ taking values in K_C, since κ is a theta characteristic.

We can generalize the above construction to families of line bundles on $\widetilde{C}$ parametrized by the Prym variety P and its coset P_1 via the Poincaré bundle, as discussed in lecture. In particular, consider the Poincaré bundle on $\widetilde{C} \times \widetilde{J}$ restricted to $\widetilde{C} \times P$ and $\widetilde{C} \times P_1$. The result of Mumford gives that the map $\widetilde{L} \mapsto h^0(\pi_*\widetilde{L} \otimes \kappa)$ mod 2 is constant on P and P_1. The lemma is established, since (by the push-pull formula)

$$h^0(\pi_*\widetilde{L} \otimes \kappa) = h^0(\pi_*(\widetilde{L} \otimes \pi^*\kappa)) = h^0(\pi_*(\widetilde{L} \otimes \tilde{\kappa})) = h^0(\widetilde{L} \otimes \tilde{\kappa}) \tag{5.3}$$

□

Step 2. There exists $\widetilde{L} \in \ker Nm$ such that $h^0(\widetilde{C}, \widetilde{L}) \neq 0$.

Proof. Let $\sum p_i$ be a divisor in the complete linear series $|K_C|$. We know that $|K_C|$ is non-empty since, by Riemann-Roch, $\dim |K_C| = h^0(C, K_C) - 1 = (h^0(C, \mathcal{O}_C) + (2g-2) - g + 1) - 1 = g - 1$, and we are assuming $g \geq 1$.

For each i, choose $\tilde{p}_i \in \pi^{-1}(p_i)$. Then

$$\begin{aligned} Nm(\mathcal{O}_{\widetilde{C}}(\sum \tilde{p}_i) \otimes \tilde{\kappa}^{-1}) &= K_C \otimes Nm(\pi^*\kappa)^{-1} \\ &= K_C \otimes (\kappa^2)^{-1} \\ &= \mathcal{O}_C. \end{aligned} \tag{5.4}$$

Therefore $\widetilde{L} = \mathcal{O}_{\widetilde{C}}(\sum \tilde{p}_i) \otimes \tilde{\kappa}^{-1} \in \ker Nm$, and in particular, $|\widetilde{L} \otimes \tilde{\kappa}|$ is non-empty, and so $\widetilde{L} \otimes \tilde{\kappa}$ has a non-trivial section, since the dimension of the space of sections of $\widetilde{L} \otimes \tilde{\kappa}$ is one greater than $\dim |\widetilde{L} \otimes \tilde{\kappa}| \geq 0$. □

Step 3. If $\widetilde{L} \in \ker Nm$ such that $h^0(\widetilde{L} \otimes \tilde{\kappa}) \neq 0$, then for a generic $\tilde{p} \in \widetilde{C}$,

$$h^0(\widetilde{L}(\tilde{p} - \iota(\tilde{p})) \otimes \tilde{\kappa}) = h^0(\widetilde{L} \otimes \tilde{\kappa}) - 1. \tag{5.5}$$

Proof. The proof of Step 3. requires the following result about base-points of complete linear systems.

Proposition 5.6. *If D is a divisor on a compact Riemann surface X (i.e., an algebraic curve), then the complete linear system $|D|$ has $p \in X$ as a base-point if* *and only if* $h^0(D) = h^0(D-p)$. *Moreover* $|D|$ *is base-point free if and only if for all* $p \in X$, $h^0(D-p) = h^0(D) - 1$.

Proof. Since $|D| = \mathbb{P}(H^0(D))$,

$$\begin{aligned} p \text{ is a basepoint for } |D| &\Leftrightarrow \forall E \in |D|, \quad p \in E \\ &\Leftrightarrow \forall f \in \mathcal{M}(X), \text{ if } (f) + D \geq 0, \text{ then } ord_p(F) + D|_p \geq 1 \\ &\Leftrightarrow H^0(D) \subseteq H^0(D-p). \end{aligned}$$

Since $H^0(D-p) \subseteq H^0(D)$, this proves the first half.

For the second claim, note that $H^0(K_X - D) \subseteq h^0(K_X - D + p)$, so we have the corresponding inequality for their respective dimensions. Then by Riemann-Roch,

$$h^0(D) - h^0(D-p) = 1 + h^0(K_X - D) - h^0(K_X - D + p) \leq 1$$

By the first claim, $|D|$ is base-point free if and only if for all $p \in X$, $h^0(D-p) < h^0(D)$, but the above calculation implies that the inequality can be no greater than 1, proving the second claim. □

To prove Equation 5.5, let $r = h^0(\widetilde{L} \otimes \tilde{\kappa})$. Choose a $\tilde{p} \in \widetilde{C}$ such that $\iota(\tilde{p})$ is not a basepoint for the complete linear system. Then by Proposition 5.6,

$$h^0(\widetilde{L}(-\iota(\tilde{p}) \otimes \tilde{\kappa})) = r - 1, \tag{5.6}$$

For notational convenience, pick a divisor D such that $[D] = \widetilde{L}(-\iota(\tilde{p})) \otimes \tilde{\kappa}$. Proposition 5.6 implies that either $h^0(K_{\widetilde{C}} - D) - h^0(K_{\widetilde{C}} - D - \tilde{p}) = 0$ or 1, depending on whether or not $\tilde{p}$ is a base-point of $|K_{\widetilde{C}} - D|$, so by Riemann-Roch, either $h^0(D + \tilde{p}) - h^0(D) = 0$ or 1, according to whether $\tilde{p}$ is, or is not, a base-point of $|K_{\widetilde{C}} - D|$. In the original notation, $h^0(\widetilde{L}(\tilde{p} - \iota(\tilde{p})) \otimes \tilde{\kappa}) = r$ or $r + 1$.

If it equals r, Riemann-Roch implies that $h^1(\widetilde{L}(\tilde{p} - \iota(\tilde{p})) \otimes \tilde{\kappa}) = h^1(\widetilde{L}(-\iota(\tilde{p})) \otimes \tilde{\kappa})$, so by Serre duality, $h^0(\widetilde{L}^{-1}(\iota(\tilde{p}) - (\tilde{p})) \otimes \tilde{\kappa}) = h^0(\widetilde{L}^{-1}(\iota(\tilde{p})) \otimes \tilde{\kappa})$, since $\tilde{\kappa} \otimes \tilde{\kappa} = K_{\widetilde{C}}$. Appealing once more to Proposition 5.6, it follows that $\tilde{p}$ is a base-point of $|\widetilde{L}^{-1}(\iota(\tilde{p})) \otimes \tilde{\kappa}|$. But since $H^0(\widetilde{L}^{-1} \otimes \tilde{\kappa}) \subseteq H^0(\widetilde{L}^{-1}(\iota(\tilde{p})) \otimes \tilde{\kappa})$, $\tilde{p}$ is also a basepoint of $|L^{-1} \otimes \tilde{\kappa}|$.

On the other hand,

$$h^0(\widetilde{L}^{-1} \otimes \tilde{\kappa}) = h^1(\widetilde{L}^{-1} \otimes \tilde{\kappa}) = h^0(\widetilde{L} \otimes \tilde{\kappa}) > 0, \tag{5.7}$$

where the first equality follows from Riemann-Roch, noting that $deg(\tilde{\kappa}) = 2g - 2$ and $deg(\widetilde{L}) = 0$, and the second equality follows from Serre duality, since $\tilde{\kappa}$ is a theta characteristic. The final inequality follows from the choice of $\widetilde{L}$ provided by Step 2.

Therefore $h^0(\widetilde{L}(\tilde{p} - \iota(\tilde{p})) \otimes \tilde{\kappa}) = r$ cannot hold for a generic $\tilde{p}$, and so for almost all $\tilde{p} \in \widetilde{C}$, $h^0(\widetilde{L}(\tilde{p} - \iota(\tilde{p})) \otimes \tilde{\kappa}) = r - 1$. Consequently, a generic $\tilde{p}$ satisfies Equation (5.5). □

Step 4. To finish the proof of Theorem 5.5, we temporarily relabel P as P_0, and claim that if $\widetilde{L} \in P_i$, then $\widetilde{L}(\tilde{p} - \iota(\tilde{p})) \in P_{1-i}$.

Proof. Note that the kernel of Nm can be written as $P'_0 \cup P'_1$, where

$$P_0' = \{\widetilde{L} \in \widetilde{J} | \widetilde{L} = \sum_{i=1}^{2N} \mathcal{O}_{\widetilde{C}}(\tilde{p}_i - \iota\tilde{p}_i) : N \geq 0, \tilde{p}_i \in \widetilde{C}\}, \text{ and}$$

$$P_1' = \{\widetilde{L} \in \widetilde{J} | \widetilde{L} = \sum_{i=1}^{2N+1} \mathcal{O}_{\widetilde{C}}(\tilde{p}_i - \iota\tilde{p}_i) : N \geq 0, \tilde{p}_i \in \widetilde{C}\}$$

We claim that $P_0' = P_0$ and $P_1' = P_1$.

P_0' is a subgroup of $\ker Nm$ and is clearly closed. If $\tilde{p} \in \widetilde{C}$ is an arbitrary point, then $P_1' = P_0 + \mathcal{O}_{\widetilde{C}}(\tilde{p} - \iota\tilde{p})$. The leftward inclusion is obvious. For the other direction, let $\widetilde{L} = \sum_{i=1}^{2N+1} \mathcal{O}_{\widetilde{C}}(\tilde{p}_i - \iota\tilde{p}_i) \in P_1'$ be arbitrary. Then

$$\widetilde{L} = \big((\sum_{i=1}^{2N+1} \mathcal{O}_{\widetilde{C}}(\tilde{p}_i - \iota\tilde{p}_i)) + \mathcal{O}_{\widetilde{C}}(\iota\tilde{p} - \iota(\iota\tilde{p})) \big) + \mathcal{O}_{\widetilde{C}}(\tilde{p} - \iota\tilde{p}),$$

so $\widetilde{L} \in P_0 + \mathcal{O}_{\widetilde{C}}(\tilde{p} - \iota\tilde{p})$, as desired. □

Hence $P_0' \cong P_1'$, since addition is a diffeomorphism of $\widetilde{J}$; in particular, P_1' is also closed. Therefore P_0' and P_1' are the connected components of $\ker Nm$, and noting that the identity is contained in P_0' proves that $P_0' = P_0$ and $P_1' = P_1$.

The claim of Step 4 is obvious for P_0' and P_1'. □

As before, we think of $W_{\tilde{g}-1} \subset \widetilde{J}_{\tilde{g}-1}$ as the canonical theta divisor. Translating with the fixed (even) theta characteristic, $\tilde{\kappa}$, we define a theta divisor $\widetilde{\Theta}$ of $\widetilde{J} = Pic^0(\widetilde{C})$ by

$$\widetilde{\Theta} = W^{\tilde{g}-1} - \tilde{\kappa}. \tag{5.8}$$

Proposition 5.7. (1) *$P_1 \subseteq \widetilde{\Theta}$;*

(2) *There exists a theta divisor Ξ defining the principal polarization of P such that $i_P^*\widetilde{\Theta} = 2\Xi$.*

Proof. First note that there is equality in the second claim, not just equivalence.

Statement (1) follows from Riemann's theorem, since if $\widetilde{L} \in P_1$, $h^0(\widetilde{C}, \widetilde{L}) = mult_L(\widetilde{\Theta}) \geq 1$, so $\widetilde{L} \in \widetilde{\Theta}$.

For (2), if $\widetilde{L} \in \widetilde{\Theta} \cap P$, then Theorem 5.5 implies that $h^0(\widetilde{C}, \widetilde{L}) \geq 2$, so $\widetilde{L}$ is a singular point of $\widetilde{\Theta}$, and $i_P^*\widetilde{\Theta}$ consists entirely of points with multiplicity ≥ 2.

Since P is Prym-Tyurin, for any theta divisor Ξ' defining the principal polarization of P, $i_P^*\widetilde{\Theta} \equiv 2\Xi'$. Since $\phi_{2\Xi'} : P \to \hat{P}$ is surjective ($\phi_{2\Xi'}$ is an isogeny), there exists $z \in P$ such that

$$\begin{aligned} \phi_{2\Xi'}(z) &= i_P^*\widetilde{\Theta} - 2\Xi' \\ &= 2t_z^*\Xi' - \Xi', \end{aligned} \tag{5.9}$$

so setting $\Xi = t^*z(\Xi')$ gives the desired theta divisor. □

Claim (2) provides the link between the theta divisor $\widetilde{\Theta}$ on $\widetilde{J}$ and the singularity locus of Ξ. By (2), every point of Ξ is of even multiplicity when considered as a point of $\widetilde{\Theta}$, so all singular points of Ξ are also singular points of $\widetilde{\Theta}$. We see in particular that all points $z \in P$ such that $mult_z(\widetilde{\Theta}) \geq 4$ are singular in Ξ.

To understand the singular points z of Ξ such that $mult_z(\widetilde{\Theta}) = 2$, we first recall the definition of the the tangent cone to an effective divisor $D \subseteq M$, where M is a complex manifold. For $p \in M$, select holomorphic coordinates $z_1, \ldots, z_n$ centered at p. If f is the defining equation for D, we can expand $f(z)$ as a series of homogeneous forms,

$$f(z) = \sum_{k=0}^{\infty}. \tag{5.10}$$

In terms of the basis $\left\{\frac{\partial}{\partial z_i}\right\}$ for T_pM, we define the *tangent cone*, TC_pD as the algebraic subvariety of T_pM defined by $f_h(z) = 0$, where h is the minimum integer such that f_h is not identically zero.

Turning again to Equation (5.3), let $\tilde{\vartheta}$ be a Taylor expansion for the theta function of $\widetilde{\Theta}$ about the point $x \in \widetilde{J}$. Then $\tilde{\vartheta}$ restricts on T_xP to ξ^2, where ξ is the Taylor expansion at x of a theta function for Ξ. Hence the lowest order term of $\tilde{\vartheta}$ not vanishing identically equals the square of the lowest non-vanishing term of ξ, i.e., equals the square of the defining function for $TC_x\Xi$.

Let $\tilde{\vartheta}_h$ be the lowest non-vanishing term of $\tilde{\vartheta}$. Then

$$\tilde{\vartheta}|_{T_xP} = \tilde{\vartheta}_h + \ldots = \xi^2 = (\xi_{h/2} + \ldots)^2, \tag{5.11}$$

and so

$$\begin{aligned} T_xP \subseteq TC_x(\widetilde{\Theta}) &\Leftrightarrow \tilde{\vartheta}|_{T_xP} = (\xi_{h/2})^2 \equiv 0 \\ &\Leftrightarrow \xi_{h/2} \text{ is not the first non-vanishing term of } \xi \\ &\Leftrightarrow mult_x(\Xi) \neq h/2. \end{aligned}$$

Therefore if $T_xP \subseteq TC_x(\widetilde{\Theta})$ and $mult_x(\widetilde{\Theta}) = 2$, $mult_x(\Xi) \geq 2$, and in particular, $x \in sing\ \Xi$, as desired.

References

[ACGH] E. Arbarello, M. Cornalba, P.A. Griffiths and J. Harris, *Geometry of Algebraic Curves*, vol. I (New York, 1985).

[BL] C. Birkenhake and H. Lange, *Complex Abelian Varieties* (New York, 2004).

[GHL] C. Gómez, R. Hernándeza and E. López, Integrability, Jacobians and Calabi-Yau threefolds, hep-th/9604057, 11 Apr 1996.

[GH] P. Griffiths and J. Harris, *Principles of Algebraic Geometry* (New York, 1978).

[Har] J. Harris, *Theta-characteristics on algebraic curves*, Trans. Am. Math. Soc. (2) **271** (1982), 611-638.

[Hart] R. Hartshorne, *Algebraic Geometry* (New York, 1977).

[Mc] C. McMullen, Prym Varieties and Teichmuller Curves, 16 March 2006.

[Mir] R. Miranda, *Algebraic Curves and Riemann Surfaces* (Providence, 1995).

[M1] D. Mumford, *Theta-characteristics on algebraic curves*, Ann. Ecole Norm. Sup (4) **4** (1971), 181-192.

[M2] D. Mumford, *Prym Varieties I*, in *Contributions to Analysis* (New York, 1974), 325-350.

[SV] R. Smith and R. Varley, *A Riemann Singularities Theorem for Prym Theta Divisors, With Applications*, Pacific Journal of Mathematics (2) **201** (2001), 479-509.