

# Torelli's Theorem

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## Section 0.

This is a homework project for a course on Abelian Varieties taught in the Spring of 2006 by G. Farkas. We will present a proof of the following result:

**Theorem (Torelli).** The necessary and sufficient condition for two curves  $X$  and  $Y$  of the same genus  $g$  to be birationally equivalent is that their symmetric product  $(X)^{(g-1)}$  and  $(Y)^{(g-1)}$  be birationally equivalent.

An immediate consequence of this is that one can recover the curve  $X$  from the theta divisor of its Jacobian variety.

We will prove the theorem by splitting the situation into two parts, namely, the non-hyperelliptic case and the hyperelliptic case. But first, we need to introduce some necessary and useful conclusions. The main content and idea of the whole proof comes from the thesis finished by Aldo Andreotti, "On a Theorem of Torelli".

## Section 1.

Now we introduce several propositions with some necessary statements here.

**Proposition 1.** If  $X$  is a projective variety of dimension  $d$ , irreducible and non-singular, the symmetric product  $(X)^{(q)}$  has a singular locus of dimension no greater than  $(q-1)d$ .

**Proposition 2.** The symmetric product  $(X)^{(g)}$  of an algebraic non-singular curve  $X$  is non-singular.

**Theorem 3.** Let  $X$  and  $Y$  be two algebraic varieties of the same dimension, both irreducible and non-singular. Let  $\phi : X \rightarrow Y$  be a birational map of  $X$  into  $Y$ , then there is an isomorphism

$$\phi^* : H^0(Y, \Omega^*) \rightarrow H^0(X, \Omega^*), \Omega^* = \Omega^{p_1} \otimes \dots \otimes \Omega^{p_r}.$$

**Proposition 4.** Let  $V$  be an irreducible affine normal variety of dimension  $d$ , let  $W$  be a subvariety of  $V$  of dimension no greater than  $d-2$ , and let  $t$  be a rational

regular function defined on  $V - W$ . The function  $t$  may be extended in a unique way to all  $V$  by a rational regular function.

**Proposition 5.** Suppose that  $X$  is of genus  $g > 1$ . The image  $\phi(X)$  of  $X$  by the canonical map is non-singular. If  $X$  is not hyperelliptic, then  $\phi : X \rightarrow C$  is biregular. If  $X$  is hyperelliptic, then the canonical map may be factored into the map  $X \rightarrow P_1(k)$  given by the subfield of genus 0 of  $k(X)$  and the biregular map of  $P_1(k)$  onto a rational normal curve of  $P_{g-1}(k)$ .

**Proposition 6.** Let  $(A_1) + \dots + (A_q)$  be the point of  $(X)^{(q)}$  corresponding to the divisor  $A_1 + \dots + A_q$  on  $X$ . Then  $\dim |A_1 + \dots + A_q| \geq 1$  iff for any holomorphic  $q$ -form  $\Omega$  on  $(X)^q$ , we have  $\Omega((A_1) + \dots + (A_q)) = 0$ .

**Proposition 7.** Let  $K$  be the field of rational functions of a complete non-singular variety  $V$  of dimension  $d$ . If  $\dim H^0(V, \Omega^{(d)}) > 0$ , then we can define the canonical image  $C$  of  $V$ . The variety  $C$  is determined with its structure of projective variety by the field  $K$ .

**Proposition 8.** Let  $C$  be the canonical image of non-hyperelliptic curve, then we have :

- 1). The tangent line  $t_p$  at the general point  $P$  of  $C$  is not tangent at any other point;
- 2). The general hyperplane cuts  $C$  in  $2g - 2$  points such that any  $g - 1$  of them are linearly independent.

**Proposition 9.** Consider for  $u \in P_{g-1}$  the set of corresponding points on  $(X)^{(g-1)}$ :

- 1). If  $X$  is not hyperelliptic, this set consists of the points on  $(X)^{(g-1)}$  corresponding to positive divisors of degree  $g - 1$  contained in the divisor cut out on  $C$  by the hyperplane  $u = 0$ .
- 2). If  $X$  is hyperelliptic, this set corresponds to positive divisors on  $X$ , which are mapped onto the divisor cut out on  $C$  by the hyperplane  $u = 0$ .

**Proposition 10.** Let  $D$  be the branch locus of the canonical map  $f$ ,

- 1). If  $X$  is not hyperelliptic,  $D$  is irreducible and a general point of  $D$  is given by the general tangent hyperplane to the canonical curve.
- 2). If  $X$  is hyperelliptic, the locus  $D$  consists of the irreducible hypersurface of the tangent hyperplanes to  $C$  and of the hyperplane which represent the stars of hyperplanes in  $P_{g-1}$  with center at the branch points on  $C$ .

**Proposition 11.** Let  $C$  be the canonical image of a non-hyperelliptic curve of genus  $g \geq 4$ . Let  $P$  be a generic point on  $C$  and let  $l_p = 0$  be the general tangent hyperplane to  $C$  at  $P$ . Then  $v_p(l_p) = 2$ .

## Section 2.

Consider the Torelli's theorem based on the characteristic  $p$ . Here we consider this by different values of  $p$ .

Case 1. For  $p = 0$ .

First we have such statement:

**Theorem.** If the symmetric product  $(X)^{(g-1)}$  and  $(Y)^{(g-1)}$  of two curves  $X$  and  $Y$  of the same genus  $g$  are birationally equivalent, then the curves  $X$  and  $Y$  are also birationally equivalent.

From the condition as above, i.e.,  $k((X)^{(g-1)}) = k((Y)^{(g-1)})$ , we have the branch locus  $D$  in  $P_{g-1}$  is the same for both symmetric product, and we can also claim that both  $X$  and  $Y$  are hyperelliptic or not. Here we discuss it separately for two cases:

1). Non-hyperelliptic;

According to the description of the locus  $D$  before, we can say that the canonical curves  $C$  and  $E$  of  $X$  and  $Y$  respectively, have the same set of tangent hyperplanes. Since now  $p = 0$ , we can claim that the plane projections of  $C$  and  $E$ , namely,  $C'$  and  $E'$  hold  $C' = E'$ . Notice that  $k(X) = k(C) = k(C') = k(E') = k(E) = k(Y)$ , here comes the result as claimed above.

2). Hyperelliptic;

Here we have  $p \neq 2$ . Notice that any hyperelliptic curve  $X$  corresponds to a quadratic extension  $k(x, y)$  of  $k(x)$ , where  $y^2 = \prod (x - a_i)$  with  $a_i \neq a_j$  for any different  $i$  and  $j$ . and it is uniquely determined by the totally  $2g + 2$  branch points  $x = a_i$ . Further we can claim that the  $2g + 2$  branch points above are able to determine the curve  $X$  because they will determine the unique normal rational curve  $C$  which contains them.

Now we consider the branch locus  $D$  again, in fact which is the same for both of two products  $(X)^{(g-1)}$  and  $(Y)^{(g-1)}$ , further, the  $D$  consists of an irreducible hypersurface of order greater than 1, representing the hyperplanes which determine in  $P_{g-1}$  the  $2g + 2$  branch points on the canonical images of  $X$  and  $Y$ . Considering the fact of the canonical image by the linear transformation over  $k_0$ , we can claim like following:

If  $X, Y$  and the birational map  $(X)^{(g-1)} \rightarrow (Y)^{(g-1)}$  are all defined over a field  $k_0 \subset k$ , then the birational map  $X \rightarrow Y$  is also defined over  $k_0$ .

And the conclusion as the theorem claimed follows by this remark.

Case 2. For  $p \neq 0$ .

Here we try to claim that the claims for  $p = 0$  will hold for any  $p \neq 2$ . Similarly, we consider this general fact under two cases :

1). Non-hyperelliptic;

The key point here is that, we need to see that if the set of tangent lines at the non-singular points of an irreducible algebraic plane curve can determine the curve uniquely.

Suppose the curve  $C: f(x, y) = 0$  to be the locus of a general point  $(x, y)$  in the affine plane, the tangent line at the point  $(x, y)$ ,  $(X - x)\partial f/\partial x + (Y - y)\partial f/\partial y = 0$ , has the coordinate  $(-x\partial f/\partial x + y\partial f/\partial y, \partial f/\partial x, \partial f/\partial y)$ , suppose that  $\partial f/\partial y \neq 0$  and two rational functions  $U = x(\partial f/\partial x)/(\partial f/\partial y) - y$  and  $V = -(\partial f/\partial x)/(\partial f/\partial y)$ , which are the coordinates of a general point of a curve  $E$ , which named as the envelop of  $C$ .

Consider the rational map  $\phi: C \rightarrow E$ , which is a map from point to point generally between  $C$  and  $E$ . We have the theorem and its corollary here:

**Theorem.** Let  $P$  be the center on  $C$  of a linear branch with tangent  $l_p = 0$ , suppose  $v_p(l_p) = q \neq 0 \pmod{p}$ . Then to this branch,  $\phi$  associates a branch on  $E$  of lines, which center at  $l_p = 0$  and with contact point on  $l_p = 0$  at point  $P$ .

**Corollary** If for a point  $P$  of  $C$ ,  $v_p(l_p) = 2$ , then for the general point  $Q$  of  $C$ ,  $v_Q(l_Q) = 2$  and if  $p \neq 2$ , this map  $\phi$  is birational.

Now we can take a general point  $Q = (x, y) = (a, \sum c_r a_r)$  in the field  $k\{\{a\}\}$  of power series with finite negative powers. If  $q = 2$ , we can see  $v_Q(l_Q) = 2$ . According to the theorem above, if for the general point  $P$  where  $v_p(l_p) = q$  and  $q \pmod{p} \neq 0$ , then the coordinates of  $P$  are rational functions of the coordinates of the general points of  $E$ .

Now we need to show that the general projection  $C'$  of the canonical image of a hyperelliptic curve  $X$  is in the condition of the above corollary.

Suppose that  $C$  is an algebraic curve in  $P_r$  and  $C'$  is its general plane projection, denote by  $P, P'$  corresponding general points on  $C$  and  $C'$  and  $l = 0$  and  $l' = 0$  for those corresponding tangent hyperplane and tangent line, considering the birational map  $: C \rightarrow C'$ , we have  $v_p(l_p) = v_{p'}(l_{p'})$ .

Further, for the genus  $g \geq 4$ , we have the Proposition 11 as list in Section 1. And we are now applying the remark as:

Let  $K = k(x)$  be the field of rational functions on an algebraic curve  $X$ , suppose that  $K = k(\phi, \psi)$  with  $[K : k(\phi)] = n, [K : k(\psi)] = m$ , then the genus of  $X$ , namely,  $g_x = nm - n - m - 1$ .

If we consider for  $X = C$  when  $m = n = 3$ , we will get the claim for  $g > 4$ . And for  $g = 4$ ,  $C$  is a curve which belongs to a surface  $F$  of second order in  $P_3$ , and we cannot have on  $C$  more than 2 series of order 3 and dimension 1. For  $g = 3$ , we have to consider only the case of  $p = 3$ , which has been proved by Gauthier.

## 2). Hyperelliptic

Under this situation, the case will be the same as the case 2) when  $p = 0$ , so nothing more need to state here.

### Section 3.

Now we try to prove the theorem in general case. As mentioned at the very beginning, we will discuss separately for two cases:

i). Non-hyperelliptic case.

Consider the curve as the non-hyperelliptic curve, and for any value of the characteristic  $p$ , according to the proposition 8 in section 2, we have such a lemma:

**Lemma.** The canonical map:  $\phi : C' \rightarrow E$  is purely inseparable.

Suppose that we have two curves  $C_1$  and  $C_2$ , with the same envelop  $E$ , how can we show that those two curves are birationally equivalent ?

Suppose that  $[k(C_1) : k(E)] = p^a$  and  $[k(C_2) : k(E)] = p^b$ , so that  $k(C_1)^{p^a} = k(E) = k(C_2)^{p^b}$ , suppose that  $j = p^a - p^b \geq 0$  then  $k(C_2) = k(C_1)^j$ , we need to show in fact  $j = 0$ , i.e.,  $a = b$ .

Let  $\sigma$  be the automorphism of the universal domain  $\Omega$  given by the exponentiation to the power  $p^{a-b}$ , now we have  $k(C_1)^\sigma = k(C_2)$ . If  $C_1$  has the equation  $f(x_1, y_1) = 0$ , then  $C_2$  will have the equation  $f^\sigma(x_2, y_2) = 0$ . If  $M_1$  is a point of  $C_1$ , then  $M_2 = M_1^\sigma$  is a point of  $C_2$ . Now  $k(E)$  is defined as the field

$$k(y_1 - x_1(\partial f / \partial x_1) / (\partial f / \partial y_1), (\partial f / \partial x_1) / (\partial f / \partial y_1))$$

applying the automorphism, we get the field:

$$k^\sigma(y_2 - x_2\{(\partial f / \partial x_1) / (\partial f / \partial y_1)\}^\sigma, \{(\partial f / \partial x_1) / (\partial f / \partial y_1)\}^\sigma)$$

Because  $k$  is perfect, we have  $k^\sigma = k$ , so  $\sigma$  carries the extension  $k(C_1)/k(E)$  over the extension  $k(C_2)/k(E)$ , which means that  $a = b$ .

ii). Hyperelliptic case.

Seems here be not a lot work remains. Because in the previous section, we have already claimed that the corresponding proof for  $p = 0$  holds without changes for any  $p \neq 2$ . So here we just need to finish the discussion for  $p = 2$ .

In the case of  $p = 2$ , we need to work on the quadratic extension  $K(y)$  of the field  $K = k(x)$  which was generated by an equation of type  $Q(x) = y^2 - y$ , where  $Q(x)$  is determined mod. a substitution of type  $Q(x) \rightarrow Q(x) + Q_0^2(x) - Q_0(x)$  with  $Q_0(x) \in k(x)$ . Further we may suppose that  $Q(x) = A(x)/B(x)$  with  $A, B$  having no common factor in  $k(x)$ . Here we make two remarks:

From Proposition 9, we can get following:

1). For a hyperelliptic curve  $X$  the model  $(X)^{(g-1)}$  is defined with respect to biregular transformation in terms of field  $k((X)^{(g-1)})$ .

From Proposition 6, we can get following:

2). The canonical system on  $(X)^{(g-1)}$  has a fixed part in the subvariety which is the locus of the point

$$(P_1) + (P_1^\sigma) + \dots + (P_{g-1}),$$

where  $P_i$  are general independent points on  $X$  and  $\sigma$  is the automorphism of  $X$  into itself given by  $g^1_2$ .

Now we consider the canonical map  $\phi : (X)^{(g-1)} \rightarrow P_{g-1}^*$ . Suppose that we have point  $y \in P_{g-1}^*$ , such that  $\phi^{-1}(y)$  has the minimum number of distinct points are given by the image of the hyperplane to the canonical image  $C$  of  $X$ . Here  $C$  is rational normal and the hyperplane at any point of  $C$  is uniquely determined and only meets  $C$  there. Therefore the locus of the point  $y$  is a rational curve  $E$  of  $P_{g-1}^*$ . Now we have the set  $\phi^{-1}(E)$ , which consist of curves  $\gamma_k$ , where  $0 \leq k \leq (g-1)/2$ , which are images on  $(X)^{(g-1)}$  of the curve  $X$  by mappings

$$f_k : Q \rightarrow k(Q) + (g-1-k)Q$$

Notice that only  $\gamma_0$  does not lie in the common part of the canonical system, also we have  $[k(X) : k(\gamma_0)] = p^\alpha$  for some nonnegative  $\alpha$ . Suppose for the curve  $Y$ , we have  $k((Y)^{(g-1)}) = k((X)^{(g-1)})$ , then we have a biregular map of  $(Y)^{(g-1)}$  to  $(X)^{(g-1)}$ , so the diagonal  $\gamma'_0$  of  $(Y)^{(g-1)}$  is birationally equivalent with the diagonal of  $(X)^{(g-1)}$ , namely,  $\gamma_0$ , further we will have  $[k(Y) : k(\gamma'_0)] = [k(X) : k(\gamma_0)]$ , which indicates that  $k(Y) \approx k(X)$ , finish the proof.