COMPACTIFICATION OF $\mathbb{A}_2$

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1. Compactification of $\mathcal{A}_2$

The construction of the moduli space of principal polarized Abelian varieties is straightforward: a g-dimensional abelian variety is given by $Z \in \mathbb{H}_g = \{Z \in M_{g\times g}(\mathbb{C}) \mid Z = {}^tZ, \Im(Z) > 0\}$, unique up to an action by the discrete group $\mathrm{Sp}(2g,\mathbb{Z})$. $\mathcal{A}_g$ is therefore the quotient $\mathbb{H}_g/\mathrm{Sp}(2g,\mathbb{Z})$. A natural problem then is to find a "nice" compactification $\mathcal{A}_g \subset \overline{\mathcal{A}_g}$ in such a way that points of the boundary $\overline{\mathcal{A}_g} \setminus \mathcal{A}_g$ correspond to degenerations of abelian varieties.

1.1. **Compactifiying $\mathcal{A}_1$.**

As a warm-up, we consider the case $g = 1$: compactifiying the moduli space of elliptic curves. There is an $\mathrm{Sl}(2,\mathbb{Z}) = \mathrm{Sp}(2,\mathbb{Z})$-equivariant function $j : \mathbb{H}_1 \to \mathbb{C}$ which identifies $\mathcal{A}_1$ with $\mathbb{C}$ and $\mathbb{C}$ has the canonical compactification $\mathbf{P}^1$. We would like to interpret the point at infinity as giving a degenerate abelian variety.

Let $D = \{z \in \mathbb{H}_1 \mid |\Re(z)| \leq \frac{1}{2},\ |z| \geq 1\}$. Under the $\mathrm{Sl}(2,\mathbb{Z})$ action, any point of $\mathbb{H}_1$ can uniquely be represented as a point of $D/\sim$ where $\sim$ represents certain gluing conditions along the boundary of D.

For $\tau \in D$, let $E_\tau = \mathbb{C}/(\mathbb{Z}\tau \oplus \mathbb{Z})$. We take the quotient in two parts. First note that $\mathbb{Z}$ acts on $\mathbb{C}$ by $n \cdot z = z + n$ and the map $z \mapsto \mathrm{e}^{2\pi i z}$ identifies $\mathbb{C}/\mathbb{Z}$ with $\mathbb{C}^*$. Then $E_\tau = \mathbb{C}^*/\mathbb{Z}\tau$ where $\mathbb{Z}\tau$ acts on $\mathbb{C}^*$ by multiplication by $t = \mathrm{e}^{2\pi i\tau}$.

Since $\tau \in D$, if τ approaches infinity, $\Im(\tau)$ approaches infinity which forces t to 0. Our elliptic curve $E_\tau = \mathbb{C}^*/\mathbb{Z}\tau$ degenerates into $\mathbb{C}^*$, the one dimensional torus, which is naturally an algebraic group. (Here and throughout, the term "torus" is reserved for the algebraic torus $(\mathbb{C}^*)^r$).

1.2. **An Example of a Degenerate Abelian Surface.**

We can of course play the same game with an abelian surface. Let $Z = \begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix} \in \mathbb{H}_2$, $\Lambda_Z = \mathbb{Z}(z_1,z_2) \oplus \mathbb{Z}(z_2,z_3) \oplus \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$ be the corresponding lattice and let $A_Z = \mathbb{C}^2/\Lambda_Z$ be the abelian surface. Again taking a two-step quotient, we obtain $A_Z = (\mathbb{C}^*)^2/\mathbb{Z}^2$. This $\mathbb{Z}^2$-action is determined by:

$$(1,0) \cdot (w_1,w_2) = (t_1 w_1, t_2 w_2) \tag{1}$$

$$(0,1) \cdot (w_1,w_2) = (t_2 w_2, t_3 w_2) \tag{2}$$

where $t_i = \mathrm{e}^{2\pi i z_i}$. By simultaneously allowing z_2, z_3 to approach infinity we get $G = (\mathbb{C}^*)^2/\mathbb{Z}(z_1, 0) = \mathbb{C}^* \times E_{z_1}$, a trivial $\mathbb{C}^*$-bundle over the elliptic curve E_{z_1}. In fact, we have

$$1 \to \mathbb{C}^* \to G \to E_{z_1} \to 1 \tag{3}$$

Definition 1.1. An algebraic group G is a **semi-abelian variety** if

$$1 \to T \to G \to A \to 1 \tag{4}$$

where T is a torus, A an abelian variety.

In both our examples abelian varieties degenerate into semi-abelian varieties. In the $g = 1$ case, this is enough to compactify $\mathcal{A}_1$. However the $g = 2$ case requires more than just $\mathbb{C}^*$-bundles over elliptic curves. This should be intuitively clear. By allowing both z_2, z_3 to approach infinity we are describing more than just a "first order" degeneration.

2. Toric Varieties

The construction of $\overline{\mathcal{A}_2}$ requires the use of toric varieties. We will also need them to describe degenerate abelian surfaces.

Definition 2.1. An r-dimensional **toric variety** is an algebraic variety X with an open embedding $T = (\mathbb{C}^*)^r \hookrightarrow X$ so that the action of T on itself extends to an action of T on X.

To construct toric varieties, we consider geometry in the lattices $M = \mathbb{Z}^r$ and $N = M^*$. M is naturally identified with $\mathrm{Hom}(T, \mathbb{C}^*)$, the group of characters of T and N with $\mathrm{Hom}(\mathbb{C}^*, T)$, the group of one-parameter subgroups of T: if $m = (m_1, \ldots, m_r) \in M$ and $n = (n_1, \ldots, n_r) \in N$,

$$m(t_1, \ldots, t_r) = t_1^{m_1} \ldots t_r^{m_r} \tag{5}$$

$$n(z) = (z^{n_1}, \ldots, z^{n_r}) \tag{6}$$

For a given lattice M, $M_{\mathbb{R}}$ will denote $M \otimes_{\mathbb{Z}} \mathbb{R}$.

We list several definitions:

Definition 2.2. A **cone** is a subset $\sigma = \mathbb{R}_{\geq 0} m_1 + \ldots + \mathbb{R}_{\geq 0} m_k$ for some $m_1, \ldots m_k \in M_{\mathbb{R}}$. If we can take the m_i's to be in M, σ is said to be a **rational polyhedral cone**. If σ does not contain a line, we say σ is **strictly convex**.

Definition 2.3. Let $\sigma \subset M$ be a cone. We define the **dual cone** $\check{\sigma}$ to be $\{n \in N \mid \langle n, m \rangle \ \geq 0 \ \forall m \in \sigma\}$.

Definition 2.4. Let σ, τ be cones in $M_{\mathbb{R}}$. τ is a **face** of σ, $\tau \prec \sigma$, if τ is the intersection of σ with a supporting hyperplane: there is some $n \in \check{\sigma}$ where $\tau = \sigma \cap n^{\perp} = \{m \in \sigma \mid \langle n, m \rangle = 0\}$.

Definition 2.5. A **fan** in $M_{\mathbb{R}}$ is a collection Σ of rational polyhedral cones such that the following two conditions hold:

(a) if $\sigma \in \Sigma$ and $\tau \prec \sigma$, $\tau \in \Sigma$.

(b) if $\sigma_1, \sigma_2 \in \Sigma$, $\sigma_1 \cap \sigma \prec \sigma_i$ for $i = 1, 2$.

All cones are assumed to be rational polyhedral cones and cones defined in $M_{\mathbb{R}}$ are assumed be strictly convex. In this case, $\check{\sigma} \subset N$ is not contained in any hyperplane and therefore has maximal dimension.

2.0.1. *Affine Toric Varieties.* Let $\sigma \subset M_{\mathbb{R}}$ be a cone. Let $\mathbb{C}[\sigma]$ be the $\mathbb{C}$-algebra generated by the formal symbols $\{x^m \mid m \in \check{\sigma} \cap N\}$ with multiplication determined by $\check{\sigma} \cap N$:

$$x^{n_1} \cdot x^{n_2} = x^{n_1 + n_2} \tag{7}$$

Since $\check{\sigma} \cap N$ is a finitely generated semi-group (as is any rational polyhedral cone), $\mathbb{C}[\sigma]$ is a finitely generated $\mathbb{C}$-algebra and hence determines an affine variety $X(\sigma) = \mathrm{Spec}(\mathbb{C}[\sigma])$.

Now suppose that $\tau \prec \sigma$, $\tau = \sigma \cap n^{\perp}$. Then $\check{\sigma} \subset \check{\tau} = < \check{\sigma}, \pm n >$. In terms of $\mathbb{C}$-algebras, $\mathbb{C}[\sigma] \hookrightarrow \mathbb{C}[\tau] = \mathbb{C}[\sigma]_{x^n}$. This realizes $X(\tau)$ as a principal open set of $X(\sigma)$, the set of points where x^n does not vanish.

Since cones are strictly convex, if $\tau = \{0\}$, $\tau \prec \sigma$ for any cone σ. $\mathbb{C}[\tau] \cong \mathbb{C}[x_1, x_1^{-1}, \ldots, x_r, x_r^{-1}]$ so that $X(\tau) = T$ is an open set of any affine toric variety.

2.0.2. *Torus Actions.* The usual action of T on itself is given in terms of a T-action on its coordinate ring $R = \mathbb{C}[x_1, x_1^{-1}, \ldots, x_r, x_r^{-1}]$. If $t = (t_1, \ldots, t_r)$ and $a_i \in \mathbb{Z}$, $t \cdot x_1^{a_1} \ldots x_r^{a_r}$ is defined to be $t_1^{a_1} \ldots t_r^{a_r} x_1^{a_1} \ldots x_r^{a_r}$. The action is then extended linearly to the rest of R.

Since the T action on R is determined by the action of T on the monomials of R and $\mathbb{C}[\sigma]$ is generated by monomials, T acts on $\mathbb{C}[\sigma]$ and hence acts on $X(\sigma)$. Furthermore, the inclusion $\mathbb{C}[\sigma] \hookrightarrow R$ is T-equivariant so that the action of T on $X(\sigma)$ extends the action of T on itself.

Example 2.6. Let M have a basis $m_1, \ldots, m_r$ and σ be the cone generated by $m_1, \ldots, m_k$. Then $\mathbb{C}[\sigma] \cong \mathbb{C}[x_1, \ldots, x_k, x_{k+1}, x_{k+1}^{-1}, \ldots, x_r, x_r^{-1}]$, $X(\sigma) = \mathbb{C}^k \times (\mathbb{C}^*)^{r-k}$. These are the only smooth affine toric varieties. We say that σ is **strictly simplicial** if σ has a set of generators which can be completed to a basis of the lattice.

Example 2.7. Let $M = \mathbb{Z}^2$ with basis m_1, m_1, σ the cone generated by m_1 and $m_1 + 2m_2$. $\check{\sigma}$ is then generated by $m_1, m_2, 2m_1 - m_2$, $\mathbb{C}[\sigma] \cong \mathbb{C}[X, Y, Z]/(X^2 = YZ)$. $X(\sigma)$ is the affine cone over a quadric in $\mathbf{P}^2$ and in particular a singular toric variety.

2.0.3. *Toric Varieties from Fans.* More generally, a toric variety is given by a fan. If Σ is a fan in M, we construct the toric variety $X(\Sigma)$ as follows. First consider $\coprod_{\sigma \in \Sigma} X(\sigma)$. If σ_1 and σ_2 are any two cones of Σ, $\tau = \sigma_1 \cap \sigma_2$

is a face of each so that $X(\tau)$ is an open set of both $X(\sigma_1)$ and $X(\sigma_2)$. We then glue $X(\sigma_1)$ and $X(\sigma_2)$ along this common open set. In other words,

$$X(\Sigma) = \coprod_{\sigma \in \Sigma} X(\sigma)/\sim \tag{8}$$

where $x_1 \sim x_2$ if $x_i \in X(\sigma_i)$ and there is a cone $\tau \subset \sigma_1 \cap \sigma_2$ such that $x_i \in X(\tau) \subset X(\sigma_i)$ and $x_1 = x_2$ in $X(\tau)$.

Example 2.8. Let $M = \mathbb{Z}^2$ with basis m_1, m_2. We let $m_3 = -m_1 - m_2$ and define $\sigma_{i,j}$ to be the cone generated by m_i, m_j. These three cones together with their faces define a fan Σ. Since any two m_i, m_j form a basis for M, $X(i,j) = X(\sigma_{i,j}) \cong \mathbb{C}^2$. These three glue together to give $X(\Sigma) = \mathbf{P}^2$.

For simplicity we glue only $X(1,2)$ with $X(1,3)$. We show that this amounts to giving coordinates $[x : y : z]$ on $\mathbf{P}^2$ and $X(1,2)$ is the open affine piece $x \neq 0$ and $X(1,3)$ is $z \neq 0$ with their usual identifications.

Note that $\sigma_{1,2}\check{}$ is generated by m_1^*, m_2^* while $\sigma_{1,3}\check{}$ is generated by $-m_2^*, m_1^* - m_2^*$. Their intersection τ is generated by m_1 and $\tau\check{}$ is generated by $m_1^*, \pm m_2^*$.

$X(1,2)$ is $\mathbb{C}^2$ with coordinates (x,y), $X(1,3)$ as $\mathbb{C}^2$ with coordinates (xy^{-1}, y^{-1}) and $X(\tau)$ as $\mathbb{C} \times \mathbb{C}^*$ with coordinates (x,y). Then the inclusion of $X(\tau)$ in $X(1,2)$ is simply $(x,y) \mapsto (x,y)$ while $X(\tau)$ in $X(1,3)$ is $(x,y) \mapsto (x/y, 1/y)$.

2.0.4. *Toric Varieties from Polytopes.* Let L be a lattice and P a polytope in $L_{\mathbb{R}}$ whose vertices lie in L. We take the cone over $P' = \{1\} \times P \subset \mathbb{R} \oplus L_{\mathbb{R}}$, $C(P) = \{(r, rx) \mid x \in P, r \geq 0\}$. $C(P) \cap \mathbb{Z} \oplus L = \{(d, dx) \mid d \in \mathbb{Z}_{\geq 0}, dx \in L, x \in P\}$ which is a finitely generated semi-group and $\mathbb{C}[P]$, the corresponding $\mathbb{C}$-algebra, is a graded ring:

$$\deg(x^{(d,dx)}) = d \tag{9}$$

We define $X(P) = \mathrm{Proj}(\mathbb{C}[S_P])$.

Example 2.9. Let $P = [0,1] \subset \mathbb{R}$. $\mathbb{C}[P]$ is generated in degree one by $x = x^{(1,0)}$ and $y = x^{(0,1)}$ so $X(P) = \mathbf{P}^1$.

Example 2.10. In a similar way we can construct $\mathbf{P}^2$ by taking the cone over the triangle in $\mathbb{R}^2$ with vertices $(0,0), (1,0), (0,1)$.

Example 2.11. $\mathbf{P}^1 \times \mathbf{P}^1$ is the cone over the square with vertices $(0,0), (1,0), (0,1), (1,1)$. $\mathbb{C}[P]$ is generated in degree one by monomials corresponding to these vertices x, y, z, w with a single relation $xw = yz$.

Note that in these two dimensional examples each edge of the polytopes give a codimension one boundary strata of $X(P) \setminus T$. For example, when $X(P) = \mathbf{P}^2$, the edges correspond to the coordinate hyperplanes. For $X(P) = \mathbf{P}^1 \times \mathbf{P}^1$, they correspond to the lines such as $\{a\} \times \mathbf{P}^1$ where $a \in \{0, \infty\}$. In general a k-dim face of P gives a k-dim boundary strata and so in particular, the torus T corresponds to the trivial face P.

3. $\overline{\mathcal{A}_2}$

We give an out of the blue compactification of $\mathcal{A}_2$ obtained by gluing two components to $\mathcal{A}_2$.

In the general case of $\mathcal{A}_g$ we consider the vector space $\mathbb{Q}^{2g}$ with its natural symplectic form. Then one looks at equivalence classes isotropic subspaces $V, \dim(V) \leq g$ where $V \sim V'$ if there is some $g \in \mathrm{Sp}(2g, \mathbb{Z})$ with $g(V) = V'$. For the $g = 2$ case we only take the class of a line and of a plane l, h with $l \subset h$. We introduce $P(V) = \mathrm{stab}(V)$ and have $P(l) \subset P(h)$ (we may assume $l \subset h$) where

$$P(l) = \{ \begin{pmatrix} \pm 1 & m & r & n \\ 0 & a & s & b \\ 0 & 0 & \pm 1 & 0 \\ 0 & c & t & d \end{pmatrix} \in \mathrm{Sp}(4, \mathbb{Z}) \mid \begin{pmatrix} a & c \\ b & d \end{pmatrix} \in \mathrm{Sl}(2, \mathbb{Z}),\ s, t \text{ determined by others}\} \tag{10}$$

$$P(h) = \{ \begin{pmatrix} {}^t B^{-1} & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} Id & C \\ 0 & Id \end{pmatrix} \mid B \in \mathrm{Gl}(2, \mathbb{Z}), {}^t C = C\} \tag{11}$$

We will quotient $\mathbb{H}_2$ by these groups, add points to this quotient to obtain varieties $Y(l), Y(h)$ and then glue these to $\mathcal{A}_2$ along suitable open sets.

3.0.5. *Y(l).* $Y(1)$ will be a codimension one boundary component. To form the quotient $\mathbb{H}_2/P(l)$ we consider

$$1 \to P'(l) \to P(l) \to P''(l) \to 1 \tag{12}$$

where elements of $P'(l)$ are of the form

$$\begin{pmatrix} 1 & 0 & r & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{13}$$

and we can consider elements of $P''(l)$ to be of the form

$$\begin{pmatrix} \pm 1 & m & n \\ 0 & a & b \\ 0 & c & d \end{pmatrix} \tag{14}$$

The "r" term in P' identifies this group with $\mathbb{Z}$ which acts on $\mathbb{H}_2$ by

$$1 \cdot \begin{pmatrix} z_1 & z_2 \\ z_2 & x_3 \end{pmatrix} = \begin{pmatrix} z_1 + 1 & z_2 \\ z_2 & x_3 \end{pmatrix} \tag{15}$$

so that

$$p : \mathbb{H}_2/P' \hookrightarrow \mathbb{C}^* \times \mathbb{C} \times \mathbb{H}_1 \tag{16}$$

$$\begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix} \mapsto (\mathrm{e}^{2\pi i z_1}, z_2, z_3) \tag{17}$$

The action of P'' descends to an action on $p(\mathbb{H}_2/P')$ which extents to an action on $\mathbb{C}\times\mathbb{C}\times\mathbb{H}_1$: P'' acts trivially on the first factor of $\mathbb{C}$ and acts on the remaining $\mathbb{C}\times\mathbb{H}_1$ by

$$\begin{pmatrix}\pm 1 & m & n\\ 0 & a & b\\ 0 & c & d\end{pmatrix}\cdot(z,\tau)=(\frac{z+m\tau+n}{c\tau+d},\frac{a\tau+b}{c\tau+d}) \tag{18}$$

Note that the action on the $\mathbb{H}_1$ factor is the usual $\mathrm{Sl}(2,\mathbb{Z})$-action which defines $\mathcal{A}_1$. In fact, there is an obvious map

$$\mathbb{C}\times\mathbb{H}_1/P''\to\mathcal{A}_1 \tag{19}$$

$$([z],[\tau])\mapsto E_{[\tau]}. \tag{20}$$

and the fiber over $[\tau]$ is identified with E_τ.

Now take the closure of $p(\mathbb{H}_2)$ in $\mathbb{C}^2\times\mathbb{H}_1$ and let $X(l)$ be the interior of this closure. Then $X(l)$ contains $\{0\}\times\mathbb{C}\times\mathbb{H}_1$ and we define

$$Y(l)=X(l)/P''(l) \tag{21}$$

We glue $p(\mathbb{H}_2)$ to $\mathcal{A}_2$, adding the component $\mathbb{C}\times\mathbb{H}_1/P''$. Thus $Y(l)$ is an open neighborhood of $\{0\}\times\mathbb{C}\times\mathbb{H}_1$ in $\overline{\mathcal{A}}_2$.

3.0.6. *P(h).* Similar to above, we write $P(h)$ as an extension

$$1\to P'(h)\to P(h)\to P''(h)\to 1 \tag{22}$$

in which elements of $P'(h)$ are of the form

$$\begin{pmatrix}Id & C\\ 0 & Id\end{pmatrix} \tag{23}$$

where $C={}^tC$ and $P''(h)\cong\mathrm{Gl}(2,\mathbb{Z})$.

$P'(h)$ acts on $\mathbb{H}_2$ as follows: If $C=\begin{pmatrix}c_1 & c_2\\ c_2 & c_3\end{pmatrix}$,

$$C\cdot\begin{pmatrix}z_1 & z_2\\ z_2 & z_3\end{pmatrix}=\begin{pmatrix}z_1+c_1 & z_2+c_2\\ z_2+c_2 & z_3+c_3\end{pmatrix} \tag{24}$$

When we quotient by $P'(h)$ we get a map

$$p_h:\mathbb{H}_2\hookrightarrow T=(\mathbb{C}^*)^3 \tag{25}$$

$$\begin{pmatrix}z_1 & z_2\\ z_2 & z_3\end{pmatrix}\mapsto(\mathrm{e}^{2\pi iz_1},\mathrm{e}^{2\pi iz_2},\mathrm{e}^{2\pi iz_3}) \tag{26}$$

The obvious thing then is to find a toric variety $X(\Sigma)$ for the torus T has a $P''(h)$-action, extending its action on $p_h(\mathbb{H}_2)$. Moreover, we would like $P''(h)$ to preserve the toric stratification of $X(\Sigma)$ (a k-dim cone in the fan corresponds to a codim k component of $X(\Sigma)\setminus T$).

In other words, we would like a $P''(h)=\mathrm{Gl}(2,\mathbb{Z})$ action on the lattice in which Σ lives that preserves the fan structure on Σ and this action on the lattice is compatible with the $P''(h)$ action on T.

To do this we consider the vector space $\mathrm{Sym}_2(\mathbb{R})$ of symmetric 2×2 matrices over $\mathbb{R}$. Let $\mathrm{Sym}_2^{\geq 0}$ be the set of all those with non-negative entries. Let

$$\sigma = \mathrm{span}_{\mathbb{R}_{\geq 0}}(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}) \tag{27}$$

We define the fan Σ to be

$$\{g(\sigma) \mid g \in \mathrm{Gl}(2, \mathbb{Z})\} \tag{28}$$

together with faces where $\mathrm{Gl}(2, \mathbb{Z})$ acts by

$$g \cdot A =^t g^{-1} A g^{-1} \tag{29}$$

All cones of this fan are strictly simplicial so that $X(\Sigma)$ is a smooth toric variety. Moreover, the union of these cones cover $\mathrm{Sym}_2^{\geq 0}$.

As before we take the closure of $p_h(\mathbb{H}_2)$ in $X(\Sigma)$ and let $X(h)$ be the interior of this closure. Then we define

$$Y(h) = X(h)/P\text{"}(h) \tag{30}$$

We glue $Y(h)$ to $\mathcal{A}_2$ in a manner analogous to above.

3.0.7. *Final Identifications.* There is overlap in the two gluing processes which we now resolve. $P(l) \subset P(h)$ and so we have a map $X(h) \to X(l)$. This is compatible with the $P''(l)$ and $P''(h)$-actions and we thus get a map

$$\pi : Y(l) \to Y(h). \tag{31}$$

Therefore we identify the points $x_l \in Y(l)$ and $x_h \in Y(h)$ whenever $\pi(x_l) = x_h$.

The following theorem is due to Alexeev and Nakamura:

Theorem 3.1. *$\overline{\mathcal{A}_2}$ is compact. Moreover, boundary points of $\overline{\mathcal{A}}_2$ represent degenerate abelian varieties.*

where $\overline{\mathcal{A}_2}$ is the variety obtained from the above gluing.

3.1. A Few Notes on Degenerate Abelian Surfaces.

Definition 3.2. A pair $(V, \mathcal{O}_V(1))$ consisting of a variety V and an ample line bundle $\mathcal{O}_V(1)$ is a **polarized stable semi-abelic variety** if

1. V is projective
2. V is semi-normal
3. there is a semi-abelian variety G acting on V with finite orbits, the stabilizer of each point of V is reduced, connected and lies in the image of $T \to G$.

These give the degenerate abelian surfaces. We will give a rough sketch in which one may construct a few of these varieties.

3.1.1. *Delaunay Decomposition.* To start, let S be a symmetric, positive definite bilinear form on a lattice L. Consider the convex hull of the set

$$\{(S(x), x) \mid x \in L\} \subset \mathbb{R} \times L_{\mathbb{R}}. \tag{32}$$

Projecting the faces of this polyhedral set gives Δ, a **Delaunay decomposition** of $L_{\mathbb{R}}$ - a paving of $L_{\mathbb{R}}$ by polytopes. (It might be instructive to draw the pictures in the case of $L = \mathbb{Z}$ and $S(x) = x^2$).

The fan Σ defined above is related to this construction: all matrices in the relative interior of a given cone lying in $\mathrm{Sym}_2^{\geq 0}$ define the same decomposition.

Example 3.3. Let S be given by the identity matrix. Then Δ is obtained by $\mathbb{Z}^2$-translates of the unit square.

Example 3.4. Let S be given by the matrix $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. Then Δ is a union of triangles, each of which is obtained by subdividing a cube from the previous example.

In both examples the decomposition is periodic with respect to translation by $\mathbb{Z}^2$.

To produce a ppsa surface, we start with a semi-abelian variety G and a decomposition of the lattice $M = \mathrm{Hom}(T, \mathbb{C}^*)$. Since we are in the $g = 2$ case we will only consider either $G = T = (\mathbb{C}^*)^2$ and A trivial or A an elliptic curve and G a $\mathbb{C}^*$-bundle over A.

3.1.2. $G = T$. These correspond to the component obtained by adding $Y(h)$ above. Here we construct two ppsa as follows. Choose a Delaunay decomposition Δ of $\mathbb{Z}^2$ which is periodic with respect to $\mathbb{Z}^2$ and let P be a polytope in Δ. Form the toric variety $X(P)$. V will be $\coprod X(P)/\sim$. Rather than define $\sim$ we will illustrate its definition with two examples.

Example 3.5. Let Δ be given by the identity matrix. Then all polytopes P are the same and $X(P) = \mathbf{P}^1 \times \mathbf{P}^1$. Note that if e is a horizontal edge of P and e' is a horizontal edge of P', we can translate P so that the image of e is e'. We do not require that the image P is P'. A similar statement holds for vertical edges. For this reason V will be $\mathbf{P}^1 \times \mathbf{P}^1/\sim$ where $\sim$ is identifying the two lines of $\mathbf{P}^1 \times \mathbf{P}^1$ corresponding to the two horizontal edges of P and identifying the two lines corresponding to the two vertical edges of P.

Example 3.6. Let Δ be given by $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. The decomposition is given by subdividing the unit cube and translating. The two triangles P, P' in the unit cube give the toric variety $\mathbf{P}^2$. In this case V will be $\mathbf{P}^2 \coprod \mathbf{P}^2/\sim$. Once we begin translating the unit cube we see that the upper half and lower half triangles are always preserved and each edge of P always lines up with the same edge of P'. For example, if we shift the picture up one, the bottom edge of P lines up with the top of edge of P'. For this reason $\sim$ is identifying each coordinate hyperplane of $X(P)$ with a unique one of $X(P')$.

3.1.3. *A an Elliptic Curve.* These will correspond to points of $Y(l)$. We let $A = E_{z_3}$, G a $\mathbb{C}*$-bundle over A (a line bundle with the 0-section removed) and $T = \mathbb{C}^*$ so that $L = \mathbb{Z}$.

$\mathbb{R}$ has a unique decomposition of $\mathbb{R}$ - intervals of length one whose endpoints are integers. The final piece of data for a ppsav in this case is given by a homomorphism $\mathbb{Z} \to \mathrm{Pic}^0(A) = A$ which is simply giving a point $[z_2] \in A$.

Now complete $\mathcal{L} \to A$ to a $\mathbf{P}^1$-bundle $\mathbf{P} \to A$. Then V will be $\mathbf{P}/\sim$ where we identify x with $x + [z_2]$. This generalizes the construction from the beginning since now we are looking only degenerating one variable; here is analogous to letting z_1 approach infinity.