

# The Poincaré Bundle

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This is a project for an abelian varieties class. We attempt to discuss the definition of the Poincaré bundle and some of its properties. Everything written here can be found in [Mum70], chapters 10-13. Many important results follow from the calculation in Corollary 0.6. In particular, Mukai's proof of the equivalence of categories between  $D(\hat{X})$  and  $D(X)$  in [Muk81], relies essentially on this calculation.

We will construct the Poincaré Bundle as a quotient of a bundle on  $X \times X$  by an action of  $K(L) \times 0$  where  $L$  is an ample line bundle on  $X$ , where  $X$  is an abelian group scheme.

First, given a line bundle  $L$  on  $X$ , define  $K(L)$  as the points  $x \in X$  such that  $T_x^*(L) \cong L$ , that is, the points for which  $L$  is translation invariant. This is a subgroup in the case of varieties. We define it differently for group schemes. Start with the line bundle

$$M = m^*(L) \otimes p_1^*(L)^{-1} \otimes p_2^*(L)^{-1}$$

on  $X \times X$ . We need the following theorem from [M], Chapter 10.

**Theorem 0.1.** *Let  $X$  be a complete variety,  $Y$  any scheme and  $L$  a line bundle on  $X \times Y$ . Then there exists a unique closed subscheme  $Y_1 \hookrightarrow Y$  having the following properties:*

1. *if  $L_1$  is the restriction of  $L$  to  $X \times Y_1$ , there is a line bundle  $M_1$  on  $Y_1$  and an isomorphism  $p_2^*(M_1) \cong L_1$  on  $X \times Y_1$ .*
2. *if  $f : Z \rightarrow Y$  is any morphism such that there exists a line bundle  $K$  on  $Z$  and an isomorphism  $p_2^*(K) \cong (1_X \times f)^*(L)$  on  $X \times Z$ ,  $f$  can be factored as  $Z \rightarrow Y_1 \hookrightarrow Y$ .*

$Y_1$  is called the maximal subscheme of  $Y$  such that  $L|_{X \times Y_1}$  is trivial.

**Remark 0.2.** Note that  $L|_{X \times Y_1}$  is not actually trivial. It's just called that.

Now we define  $K(L)$  to be the maximal subscheme of  $X$  such that  $M|_{K(L) \times X}$  is trivial.

Given  $f \in X(S)$ , an  $S$ -valued point of  $X$ , define  $X_S = X \times S$  and let  $T_f : X_S \rightarrow X_S$  be defined as follows:

$$(1) \quad \begin{array}{ccccc} & & X \times S & & \\ & \swarrow^{1_X \times f} & \downarrow^{T_f} & \searrow^{p_2} & \\ X \times X & & X \times S & & \\ \downarrow^m & \swarrow^{p_1} & & \searrow^{p_2} & \\ X & & & & S \end{array}$$

Let  $L_S$  be the induced line bundle  $p_1^*(L)$  on  $X_S$ . Consider the condition (\*):

$$T_f^*(L_S) \cong L_S \otimes p_2^*(N), \quad N \text{ a line bundle on } S.$$

**Claim 0.3.** (\*) holds if and only if  $f$  is an  $S$  valued point of  $K(L)$ , i.e.  $f$  factors through  $K(L)$ .

*Proof.* First, we can see that  $L_S|_{(0) \times S}$  is trivial by considering the following commutative diagram:

$$(2) \quad \begin{array}{ccccc} & & (0) \times S & & \\ & \swarrow^{p_1} & \downarrow^{i \times 1_S} & \searrow^{p_2} & \\ (0) & & X \times S & & \\ \downarrow^i & \swarrow^{p_1} & & \searrow^{p_2} & \\ X & & & & S \end{array}$$

Thus  $L_S|_{(0) \times S} \cong (i \times 1_S)^* p_1^*(L) \cong p_1^* i^*(L)$  which is trivial since  $i^*(L)$  to  $(0)$  is clearly trivial.

From the diagram (1) above it is easy to see that the composite

$$X \times S \xrightarrow{T_f} X \times S \xrightarrow{p_1} X$$

is exactly the composite

$$X \times S \xrightarrow{1_X \times f} X \times X \xrightarrow{m} X$$

so  $T_f^*(L_S) \cong (1_X \times f)^* m^*(L)$ . We want to compare  $L_S|_{(0) \times S}$  with  $T_f^*(L_S)|_{(0) \times S} \cong i^* T_f^* p_1^*(L) \cong i^*(1_X \times f)^* m^*(L)$ . Now the following diagram

$$\begin{array}{ccccc}
 & & (0) \times S & & \\
 & p_1 \swarrow & \downarrow i \times 1_S & \searrow p_2 & \\
 (0) & & X \times S & & \\
 \downarrow i & p_1 \swarrow & \downarrow 1_X \times f & \searrow p_2 & \\
 X & & X \times X & & S \\
 \downarrow 1_X & p_1 \swarrow & \downarrow m & \searrow p_2 & \downarrow f \\
 X & & X & & X
 \end{array}$$

is the same as

$$\begin{array}{ccccc}
 & & (0) \times S & & \\
 & p_1 \swarrow & \downarrow i \times f & \searrow p_2 & \\
 (0) & & X \times X & & S \\
 \downarrow i & p_1 \swarrow & \downarrow m & \searrow p_2 & \downarrow f \\
 X & & X & & X
 \end{array}$$

which is the same as

$$\begin{array}{c}
 (0) \times S \\
 \downarrow 1_{(0)} \times f \\
 (0) \times X \\
 \downarrow i \times 1_X \\
 X \times X \\
 \downarrow m \\
 X
 \end{array}$$

But by the properties of group schemes,  $m \circ (i \times 1_X)$  is exactly the canonical isomorphism of  $(0) \times X$  with  $X$ , and identifying  $S$  with  $(0) \times S$  we get

$$i^*(1_X \times f)^* m^*(L) \cong f^* L.$$

In summary,  $L_S|_{(0) \times S}$  is trivial, while  $T_f^*(L_S)|_{(0) \times S} \cong f^*L$ .

Now if (\*) holds, then  $T_f^*(L_S) \cong L_S \otimes p_2^*N$ , and restricting both sides to  $(0) \times S$  we see that

$$f^*(L) \cong T_f^*(L_S)|_{(0) \times S} \cong L_S \otimes p_2^*N|_{(0) \times S} \cong N.$$

Thus (\*) holds if and only if

$$(1_X \times f)^*m^*L \cong p_1^*L \otimes p_2^*(f^*L).$$

Thus (\*) holds if and only if  $(1_X \times f)^*M$  is trivial.

Thus by the theorem above,  $f$  can be factored through  $K(L)$ , proving the claim.  $\square$

Thus for any scheme  $S$ , the set  $K(L)(S)$  is a subgroup of  $X(S)$ , and thus the subscheme  $K(L)$  is a subgroup scheme of  $X$ .

Using the fact that  $K(L)$  is a subgroup scheme of  $X$  we proceed to define  $\widehat{X}$ , the dual abelian variety to  $X$  and  $P$ , the Poincaré line bundle on the product  $\widehat{X} \times X$ . Let  $L$  be an ample line bundle on  $X$ . As we did in class, or it can be found on [Mum70], pg. 60, since  $L$  is ample on  $X$ ,  $K(L)$  is a finite group scheme. Since we will use it often we recall the following theorem about finite group schemes acting on a scheme.

**Theorem 0.4 ([Mum70], pg. 111).** *(A) Let  $G$  be a finite group scheme acting on a scheme  $X$  such that the orbit of any point is contained in an affine open subset of  $X$ . Then there is a pair  $(Y, \pi)$  where  $Y$  is a scheme and  $\pi : X \rightarrow Y$  a morphism, satisfying the following conditions:*

1. *as a topological space,  $(Y, \pi)$  is the quotient of  $X$  for the action of the underlying finite group;*
2. *the morphism  $\pi : X \rightarrow Y$  is  $G$ -invariant, and if  $\pi_*(\mathcal{O}_X)^G$  denotes the subsheaf of  $\pi_*(\mathcal{O}_X)$  of  $G$ -invariant functions, the natural homomorphism  $\mathcal{O}_Y \rightarrow \pi_*(\mathcal{O}_X)^G$  is an isomorphism.*

*The pair  $(Y, \pi)$  is uniquely determined up to isomorphism by these conditions. The morphism  $\pi$  is finite and surjective.  $Y$  will be denoted  $X/G$ , and it has the functorial property: for all  $G$ -invariant morphisms  $f : X \rightarrow Z$ , there exists a unique morphism  $g : Y \rightarrow Z$  so that  $f = g \circ \pi$ .*

(B) Suppose further that the action of  $G$  is free and  $G = \text{Spec}(R)$ ,  $n = \dim_k R$ . Then  $\pi$  is a flat morphism of degree  $n$ , i.e.  $\pi_*(\mathcal{O}_X)$  is a locally free  $\mathcal{O}_Y$ -module of rank  $n$ , and the subscheme of  $X \times X$  defined by the closed immersion

$$(\mu, p_2) : G \times X \rightarrow X \times X$$

is equal to the subscheme  $X \times_Y X \subset X \times X$ . Finally, if  $\mathcal{F}$  is a coherent  $\mathcal{O}_Y$ -module,  $\pi^*\mathcal{F}$  has a naturally defined  $G$ -action lifting that on  $X$ , and

$$\mathcal{F} \mapsto \pi^*\mathcal{F}$$

is an equivalence of the category of coherent  $\mathcal{O}_Y$ -modules (resp. locally free  $\mathcal{O}_Y$ -modules of finite rank) and the category of coherent  $\mathcal{O}_X$ -modules with  $G$ -action (resp. locally free  $\mathcal{O}_X$ -modules of finite rank with  $G$ -action).

As Mumford remarks on [Mum70], pg. 69, the condition that any  $G$ -orbit be contained in an affine open is always satisfied when  $X$  is quasi-projective. Therefore, with this theorem in mind, we can define the dual abelian variety  $\widehat{X}$  to be the quotient abelian variety  $X/K(L)$ , and let  $\pi : X \rightarrow \widehat{X}$  be the natural morphism. We can now begin to define the Poincaré line bundle,  $P$ , on  $\widehat{X} \times X$ . Following what was done in class or in the characteristic 0 case in [Mum70], pg. 78, the Poincaré bundle  $P$  on  $\widehat{X} \times X$  is defined by specifying an action of  $K(L)$  on the line bundle  $M$  on  $X \times X$  and defining  $P$  to be the equivalent line bundle on the quotient  $\widehat{X} \times X$ , which uniquely exists, up to isomorphism, by Theorem 0.4. The action of  $K(L)$  on  $M$  will lift the action of translation by  $K(L) \times \{0\}$  on  $X \times X$ .

In order to carry this out we need to specify what we mean by an action of  $K(L)$  on  $M$  which lifts the action of translation by  $K(L) \times \{0\}$  on  $X \times X$ . As in [Mum70], page 111, we define a *lift* of the action  $m : (K(L) \times \{0\}) \times (X \times X) \rightarrow (X \times X)$  to  $M$  by specifying, for every  $S$ -valued point  $f \in \underline{K(L)}(S)$ , an automorphism  $\lambda_f$  of the sheaf  $M_S = M \otimes \mathcal{O}_S$  on  $(X \times X)_S = X_S \times X_S$  covering the automorphism  $X_S \times X_S \xrightarrow{m_f} X_S \times X_S$  where  $m_f$  is the map given by  $(m \circ (1 \times f), p_2)$ . In particular, we need to construct isomorphisms  $T_{(x,0)}^*(M_S) \cong M_S$  for each  $x \in \underline{K(L)}(S)$  which makes the action of  $K(L) \times \{0\}$  on  $M_S$  into a group action which lifts the action  $X_S \times X_S \xrightarrow{T_{(x,0)}} X_S \times X_S$ , where the map  $T_{(x,0)}$  is given below.

Following [Mum70], pg. 124, we will denote all objects which are obtained by base extension to  $S$  by a subscript  $S$ . Let  $T_x : X_S \rightarrow X_S$  denote translation

by  $x$  and let  $T_{(x,0)} : X_S \times X_S \rightarrow X_S \times X_S$  denote translation by  $(x, 0)$ . From the proof above, all elements  $x \in \underline{K(L)}(S)$  satisfy  $T_x^*(L_S) \cong L_S \otimes L_0$ , where  $L_0$  is the lift to  $X_S$  of a line bundle  $\overline{L_0}$  on  $S$ . Extending the base from  $X \times X$  to  $X_S \times X_S = (X \times X)_S$ , we see that by definition

$$M_S \cong m_S^*(L_S) \otimes p_1^*(L_S)^{-1} \otimes p_2^*(L_S)^{-1}.$$

The following calculation shows that  $T_{(x,0)}^*(M_S) \cong M_S$ , so that defining a group action is possible, we will just have to choose the right isomorphisms.

$$\begin{aligned} T_{(x,0)}^*(M_S) &\cong T_{(x,0)}^*m_S^*(L_S) \otimes T_{(x,0)}^*p_1^*(L_S)^{-1} \otimes T_{(x,0)}^*p_2^*(L_S)^{-1} \\ &\cong m_S^*T_x^*(L_S) \otimes p_1^*T_x^*(L_S)^{-1} \otimes p_2^*(L_S)^{-1} \\ &\cong m_S^*(L_S \otimes L_0) \otimes p_1^*(L_S \otimes L_0)^{-1} \otimes p_2^*(L_S)^{-1} \\ &\cong m_S^*(L_S) \otimes m_S^*(L_0) \otimes p_1^*(L_S)^{-1} \otimes m_S^*(L_0)^{-1} \otimes p_2^*(L_S)^{-1} \\ &\cong M_S \otimes m_S^*(L_0) \otimes p_1^*(L_0)^{-1} \\ &\cong M_S \end{aligned}$$

The last isomorphism follows because  $L_0$  is the pullback of a bundle on  $S$ . Now that we have that the bundles  $M_S$  and  $T_{(x,0)}^*(M_S)$  are isomorphic for all  $x \in \underline{K(L)}(S)$ , we still have to show that we can choose the right isomorphism for each  $x \in \underline{K(L)}(S)$  which makes the action of  $\underline{K(L)}(S)$  into a group action. We present here an outline of the argument showing this is possible. As in the characteristic 0 case, which is given in [Mum70], pg. 79, it suffices to give our isomorphism of  $M_S$  and  $T_{(x,0)}^*(M_S)$  on the subscheme  $X_S \times 0_S$ . Let  $V \times M_S$  be the trivial line bundle over  $X_S$  with fibre  $V$ , where  $V$  is the 1-dimensional vector space dual to the fiber  $L/\mathcal{M}_0L$  of  $L$  at 0 on  $X$ . Then, if  $i : X_S = X_S \times 0_S \rightarrow X_S \times X_S$  is the closed immersion one can show that  $i^*(M_S) \cong V \times M_S$ . One then chooses the group action of  $K(L)$  on  $M$  by specifying that on the subscheme  $X_S \times 0_S$ , the lift is the map  $1_V \times T_x$  for each  $x \in \underline{K(L)}(S)$ . One needs to prove that this makes the action of  $\underline{K(L)}(S)$  on  $M_S$  into a group action.

Using this lift of the action and Theorem 0.4, there is a unique line bundle  $P$  on  $\widehat{X} \times X$  whose pull back to  $X \times X$  as a line bundle acted on by  $K(L)$  is isomorphic to  $M$  acted on by  $K(L)$  in this prescribed way. Now, what makes  $P$  a ‘‘Poincaré’’ bundle is that it satisfies the following characteristics. First of all,  $P|_{\{0\} \times X}$  is the trivial bundle and, as an abstract group, there is an isomorphism  $\widehat{X} \cong \text{Pic}^0 X$ . Recall  $\text{Pic}^0 X = \ker(\phi : \text{Pic} X \rightarrow \text{Hom}(X, \text{Pic} X))$ , where  $\phi(L) = \phi_L$  is the map defined by  $\phi_L(x) = \text{iso. class of } T_x^*L \otimes L^{-1}$  in  $\text{Pic}(X)$ .

The isomorphism  $\widehat{X} \cong \text{Pic}^0 X$  satisfies the property that for all  $\alpha \in \widehat{X}$ , the restriction of  $P$  to  $\{\alpha\} \times X$ , considered as a line bundle on  $X$ , is the element of  $\text{Pic}^0 X$  corresponding to  $\alpha$ . To complete the argument that  $P$  is a ‘‘Poincaré’’ bundle, according to Mumford’s description of the characteristics of a Poincaré bundle on page 78 of [Mum70], he proves the following theorem.

**Theorem 0.5** ([Mum70], pg. 125). *Let  $S$  be any scheme and  $L$  a line bundle on  $S \times X$  so that  $L|_{S \times \{0\}}$  is trivial and  $L|_{s \times X} \in \text{Pic}^0 X$  for all  $s \in S$ . Then there exists a unique morphism  $\phi : S \rightarrow \widehat{X}$  so that  $L \cong (\phi \times 1_X)^*(P)$ .*

The original aim of this project was to compute the cohomology of the Poincaré bundle  $P$  on  $\widehat{X} \times X$ . The result is:

**Corollary 0.6** ([Mum70], page 129).

$$H^i(\widehat{X} \times X, P) = \begin{cases} (0) & \text{if } i \neq g \\ k & \text{if } i = g. \end{cases}$$

*Outline of proof:* This corollary is proven by showing that one can use the Koszul complex of the residue field  $k$  of the regular local ring  $\mathcal{O}_{0, \widehat{X}}$  to compute the cohomology of  $P$ . In particular, using the base change theorem for direct images ([Mum70], pg. 46), Mumford gets the existence of

$$K_\bullet : 0 \rightarrow K_0 \rightarrow K_1 \rightarrow \dots \rightarrow K_g \rightarrow 0,$$

a complex of free modules of finite type over the local ring  $\mathcal{O}_{0, \widehat{X}}$ . He then proves, using a lemma stating that cohomology is often zero when you are working with Artinian modules ([Mum70], pg. 127), that there is an exact sequence

$$0 \rightarrow K_0 \rightarrow K_1 \rightarrow \dots \rightarrow K_g \rightarrow H^g(\widehat{X} \times X, P) \rightarrow 0.$$

Taking the dual of this sequence, Mumford proves that

$$0 \rightarrow \widehat{K}_g \rightarrow \widehat{K}_{g-1} \rightarrow \dots \rightarrow \widehat{K}_0 \rightarrow k \rightarrow 0$$

is a free resolution of  $k$ . Now, since any two free resolutions of  $k$  will be homotopically equivalent, we can use the Koszul complex mentioned above. Taking the dual of the Koszul complex, which Mumford notes is isomorphic to the original Koszul complex, one has a concrete way of realizing  $H^g(\widehat{X} \times X, P)$  as  $k$ .

## References

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- [Mum70] David Mumford. *Abelian varieties*. Tata Institute of Fundamental Research Studies in Mathematics, No. 5. Published for the Tata Institute of Fundamental Research, Bombay, 1970.