Algebraic Geometry

1 3/7/05- Notes taken by Kevin Klonoff

Homework review.

1.1 Veronese Embedding

For each d the veronese embedding is defined as follows:

$$\partial_d : \mathbb{P}^m \to \mathbb{P}^N$$

where $N = \begin{pmatrix} n+d \\ d \end{pmatrix} - 1$, is given by

$$\partial_d[x_0 : , \dots, : x_m] = [z_0 : , \dots, : z_N]$$

with $z_i = x_0^{i_0} x_1^{i_1} \dots x_m^{i_m}, i_0 + \dots + i_m = d$.
In other words $\partial_d[x_0, \dots, x_m] =$ [monomials in x_i of degree d].
Examples.
$m = 1$, $d = 2$.

$$\partial_2 : \mathbb{P}^1 \to \mathbb{P}^2$$

$$\partial_2[x, y] = [x^2 : xy : y^2].$$

The image inside $\mathbb{P}^2$ is a conic.
$m = 1$, $d = 3$.

$$\partial_3 : \mathbb{P}^1 \to \mathbb{P}^3$$

$$\partial_3[x : y] = [x^3 : x^2y : xy^2 : y^3].$$

The image is the projectivized twisted cubic. Recall the twisted cubic is defined by as follows:

Let $[x_0 : x_1 : x_2 : x_3]$ be the coordinates on $\mathbb{P}^3$. Then the twisted cubic is defined as $\det_2 \begin{vmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{vmatrix} = 0$ ($\det_2$ refers to the 2×2 minors). The twisted cubic is cut out by quadrics.

The Veronese surface.

$$\partial_2 : \mathbb{P}^2 \to \mathbb{P}^5$$

$$\partial_2[x, y, z] = [x^2 : y^2 : z^2 : xy : yz : xz].$$

or more generally

$$\partial_2 : \mathbb{P}^n \to \mathbb{P}^{\frac{(n+1)(n+2)}{2} - 1}$$

Let $[z_{i,j}] 1 \leq j \leq j \leq n$ be coordinates on $\mathbb{P}^{\frac{(n+1)(n+2)}{2} - 1}$.

$\mathfrak{I}(\partial_2(\mathbb{P}^n))$, the ideal of the image, is given by the 2×2 minors of the $(n+1) \times (n+1)$ symmetric matrix with entries $z_{i-1,j-1}$ on the (i, j)th position.

$$A = \begin{vmatrix} z_{00} & z_{01} & z_{02} & \dots & z_{0n} \\ z_{10} & z_{11} & z_{12} & \dots & z_{1n} \\ z_{20} & z_{21} & z_{22} & \dots & z_{2n} \\ \dots & & & \ddots & 0 \\ z_{n0} & & & & z_{nn} \end{vmatrix}$$

In other words $\mathfrak{I}(\partial_2(\mathbb{P}^n))$ is generated by $\det_2(A)$.
Lets go back to the case of the veronese surface in $\mathbb{P}^5$.

$$\partial_2 : \mathbb{P}^2 \longrightarrow \mathbb{P}^5$$

$$\partial_2[x, y, z] = [x^2 : y^2 : z^2 : xy : yz : xz] = [z_0 : \cdots : z_5].$$

The equation of the surface is given by

$$\det_2 \begin{vmatrix} z_0 & z_1 & z_2 \\ z_1 & z_3 & z_4 \\ z_2 & z_4 & z_{22} \end{vmatrix} = 0.$$

The next question could be to describe explicitely the image $\partial_2(\mathbb{P}^n) = X \subset \mathbb{P}^N$. That is

1. We found generators of the ideal
2. Find the space of all relations with polynomial coefficients between the quadrics (called the sysygies of the quadrics).

Next, let $X_d \subset \mathbb{P}^n$ a hypersurface of degree d. Now map it $\partial_d(X_d) \subset \mathbb{P}^N$ becomes a hyperplane section of $\partial_d(\mathbb{P}^N)$.
That is hypersurfaces of degree d in $\mathbb{P}^n$ become linear sections of $\partial(\mathbb{P}^n) \hookrightarrow \mathbb{P}^N$.
Application.

Theorem 1.1. *Let $X_d \subset \mathbb{P}^n$ be a hypersurface of degree d. Then $\mathbb{P}^n - X_d$ is affine.*

Proof. Use the fact that $\partial_d(\mathbb{P}^n - X_d)$ is a hyperplane complement inside of $\partial_d(\mathbb{P}^n)$. □

This is a very useful technique to prove that something is affine.

1.2 Grassmannians by Rohit Ghosh

$\mathbb{G}(1, n)$. We want to show that $\mathbb{G}(1, n)$ is a projective variety, and we'll do this by showing it is a closed subset of some $\mathbb{P}^N$. Define:

$$\phi : \{\text{Lines in } \mathbb{P}^n\} \longrightarrow \mathbb{P}^N,\ N = \begin{pmatrix} n+1 \\ 2 \end{pmatrix} - 1$$

Coordinates on $\mathbb{P}^N$, $(p_{0,1}, p_{0,2}, \dots, p_{n-1,n})$.

Let $\mathcal{L}$ be line in $\mathbb{P}^n$ and let $a = [a_0, \dots, a_n]$ and $b = [b_0, \dots b_n]$ be two different points on this line.

Then map is given by

$$A = \begin{pmatrix} a_0 & \dots & a_n \\ b_0 & \dots & b_n \end{pmatrix} \to [\det_2(A)]$$

where $\det_2(A)$ denotes the 2×2 minors of A.

We need to check that this map is well defined. Any other two points on $\mathcal{L}$ can be written as $\lambda_1 a + \mu_1 b$ and $\lambda_2 a + \mu_2 b$.

The image of

$$\overline{A} = \begin{pmatrix} \lambda_1 a_0 + \mu_1 b_0 & \dots & \lambda_1 a_n + \mu_1 b_n \\ \lambda_2 a_0 + \mu_2 b_0 & \dots & \lambda_2 a_n + \mu_2 b_n \end{pmatrix} \to [\det_2(\overline{A})] = [(\lambda_1\mu_2 - \lambda_2\mu_1)(\det_2(A))] = [\det_2(A)]$$

and the mapping is indeed well defined.

We will see about injectivity later.

If $i_1 \leq j_1 \leq j_2 \leq j_3 \in \{0, \dots n\}$ we claim that the image of of $\mathbb{G}(1,n) \subset \mathbb{P}^N$ is the zero locus of the polynomials

$$-p_{i_1,j_1}p_{j_2,j_3} + p_{i_1,j_2}p_{j_1,j_3} - p_{i_1,j_3}p_{j_1,j_2} = 0$$

There are $\begin{pmatrix} n+1 \\ 4 \end{pmatrix}$ such polynomials.

Lets call the ideal generated by these polynomials $\mathfrak{p}$. It is clear that the image of this map is contained in the zeros of this ideal. So let $p \in Z(\mathfrak{p})$. We can assume $p_{01} = 1$. Then the line in $\mathbb{P}^n$ defined by the two points in $\mathbb{P}^n$, $[1 : 0 : -p_{12} : \dots : -p_{1n}]$ and $[0 : 1 : p_{02} : \dots : p_{0n}]$ maps to this point.

Note that $\mathbb{G}(1,n) \cong G(2, n+1)$ and in general if

$$G(k,n) = \{V \subset k^n : dimV = k\}$$

and this space is canonically isomorphic to $\mathbb{G}(k-1, n-1)$.

What we just saw is a special case of the Plücker embedding.

Define a map $\phi : G(k,n) \to \mathbb{P}(\Lambda^k(V))$ as follows. Let $V \subset k^n$ be a k dimensional subspace. Choose any basis of V say $(v_1, \dots v_k)$. Then

$$\phi(V) = [v_1 \wedge \dots \wedge v_k] \in \mathbb{P}(\Lambda^k(V))$$

It is easy to see that this map is well defined. If we set $w = v_1 \wedge \dots \wedge v_k$ we can recover the subspace V from a point in the image of ϕ as the set of all v such that $v \wedge w = 0$. This says the map ϕ is injective (as was claimed earlier).

Hence, we can identify $G(k,n) \subset \mathbb{P}(\Lambda^k(V))$ as the space of totally decomposable vectors (can be written as $v_1 \wedge \dots \wedge v_k$ for some vectors v_i).

Lets get specific and suppose that $k = 2$.

Now $w \in \Lambda^2(V)$ is totally decomposable if and only if $w \wedge w = 0$. If we choose a basis for V say $\{e_i\}$, then set $p_{ij} = e_i \wedge e_j$ then the equation $w \wedge w = 0$ are exactly the quadratic equations we wrote earlier for describing the image of $\mathbb{G}(1,n) \subset \mathbb{P}^N = \mathbb{P}(\Lambda^2(V))$.

1.3 Blowups

Setup.

$$Y \subset X.$$
$$Bl_Y(X) = \tilde{X} \text{ (blowup of } Y \text{ with respect to } X).$$
$$\pi : \tilde{X} \to X \text{ is a birational regular map such that } \pi_{|\tilde{X}-\pi^{-1}(Y)} \to X - Y \text{ is an isomorphism.}$$

Fibers of π over Y will be positive dimensional (In fact will be projective spaces).

General Construction:

$X \subset \mathbb{A}^n$, $Y \subset X$, $\mathfrak{I}(Y) = (f_0, \ldots, f_r)$, $f_i \in k[X_1, \ldots, X_n]$. We have a rational map

$$\phi : \mathbb{A}^n \to \mathbb{P}^r$$

given by

$$\phi(x) = [f_0(x), \ldots, f_r(x)].$$

If $U = X - Y$ then ϕ is regular on U.

Set

$$\Gamma = \{(p, \phi(p)) : p \in U\} \subset U \times \mathbb{P}^r \subset X \times \mathbb{P}^r.$$

the graph of ϕ.

Let π_1 and π_2 be the projections on $X \times \mathbb{P}^r$ onto the first and second factors respectively. $\pi_1 = \pi : \Gamma \to U$ is an isomorphism.

Next, we'll define $\tilde{X}$ to be the closure of $\Gamma \subset X \times \mathbb{P}^r$.

$$\begin{array}{ccccc}
\Gamma = \pi^{-1}(U) & \longrightarrow & \tilde{X} \subset X \times \mathbb{P}^r & & \\
\downarrow & & \downarrow{\scriptstyle \pi_1 = \pi} & \searrow^{\pi_2} & \\
U & \longrightarrow & X & & \mathbb{P}^r
\end{array}$$

Thus, we have constructed the required map $\pi : \tilde{X} \to X$ such that $\pi_{|\pi^{-1}(U)} : \pi^{-1}(U) \to U$ is an isomorphism.

$\tilde{X} = Bl_Y(X)$ the blowup of X along the locus $(f_0, \ldots, f_r)$.

Remarks:

$\tilde{X}$ is birational to X, in particular, it has the same dimension.

Example 1. *$X = \mathbb{A}^2$, $Y = 0$. $f_0 = x, f_1 = y$.*

$$\phi : \mathbb{A}^2 \to \mathbb{P}^1$$
$$(x, y) \longmapsto [f_0(x, y) : f_1(x, y)] = [x : y].$$

So ϕ is really defined on $\mathbb{A}^2 - \{0\} = U$.

$$\Gamma = \{\big((x, y), [x : y]\big) : (x, y) \neq (0, 0) \subset U \times \mathbb{P}^1\}$$

$$= \{\big((x,y),[u:v]\big) : xv - yu = 0, (x,y) \neq (0,0) \subset \mathbb{A}^2 \times \mathbb{P}^1\}.$$

Take the closure of Γ *in* $\mathbb{A}^2 \times \mathbb{P}^1$. *Then*

$$\begin{array}{c} \tilde{X} = \{\big((x,y),[u:v]\big) : xv - yu = 0\} \subset \mathbb{A}^2 \times \mathbb{P}^1 \\ \Big\downarrow \pi \\ X \end{array}$$

$$\begin{aligned} \tilde{X} = \bar{\Gamma} &\subset \{\big((x,y),[u:v] : xv - yu = 0\} \\ &= \Gamma \cup \{(0,0) \times \mathbb{P}^1\}. \end{aligned}$$

Does it happen that every $\big((0,0),[u:v]\big) \in \bar{\Gamma}$*?*

Define $\chi : \mathbb{A}^1 \to \mathbb{A}^2 \times \mathbb{P}^1$ *by*

$$t \mapsto (tu, tv, [u:v])$$

$$\chi(\mathbb{A}^1 - \{0\}) \subset \Gamma$$

$$\chi(0) = \big((0,0) \times [u:v]\big)$$

So

$$\begin{array}{cl} \tilde{X} & = Bl_{(0,0)}(\mathbb{A}^2) = \{(x,y) \times [u:v] : xv = yu\} \subset \mathbb{A}^2 \times \mathbb{P}^1 \\ \Big\downarrow \pi & \\ \mathbb{A}^2 & \end{array}$$

$$\pi^{-1}\big((x,y)\big) = (x,y) \times [x:y]$$

$$\pi^{-1}(0) = \mathbb{P}^1$$

We can think of the blowup of $\mathbb{A}^2$ *at the origin as* $\mathbb{A}^2 - \{(0,0)\} \cup \mathbb{P}^1$ *with the* $\mathbb{P}^1$ *located at the origin.*

Remark 1.2. *As a final remark, consider the following situation:*

$$\begin{array}{ccc} U & \subset & X \\ & \searrow^{\phi} & \\ & & \mathbb{P}^r \end{array}$$

$$\phi = [f_0 : \cdots : f_r]$$

ϕ *cannot, in general, be extended to* X. *But we have the following situation:*

$$\begin{array}{ccccc} X & \subset & \tilde{X} & \subset & X \times \mathbb{P}^r \\ & \swarrow & & \searrow & \\ & X & & \mathbb{P}^r & \end{array}$$

by construction ϕ *can be extended from* $\tilde{X}$ *to* $\mathbb{P}^r$. *Call it* $\tilde{\phi}$ *and* $\tilde{\phi_{|U}} = \phi$. *(Universal procedure for extending rational maps).*