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Jan. 28, Friday (Notes taken by Paul Fili)

1. Sheaves (cont.)

Last time we defined a *presheaf* on a topological space X which assigned to each open set $U \subseteq X$ a space of sections $\mathcal{F}(W) = \Gamma(U, \mathcal{F})$, along with restriction maps for open subsets $V \subseteq U \subset X$,

$$\mathrm{res}_{V,U} : \mathcal{F}(U) \to \mathcal{F}(V)$$

satisfying the usual compatibility condition. Let us now consider an example.

Example 1.1. Let $X = \mathbb{R}^n$ with the euclidean topology and define

$$\mathcal{F}(U) = C(U, \mathbb{R}) := \{f : U \to \mathbb{R} \ : \ f \text{ is continuous}\},$$

which is the (pre)sheaf of continuous real-valued functions on X.

We can now define a *sheaf*:

Definition 1.2. A presheaf $\mathcal{F}$ on X is a *sheaf* if the following gluing condition is satisfied: if $U \subseteq X$ is an open set and $\{U_\alpha\}_\alpha$ an open cover of U and $\{s_\alpha \in \mathcal{F}(U_\alpha)\}_\alpha$ a collection of elements such that

$$s_\alpha|_{U_\alpha \cap U_\beta} = s_\beta|_{U_\alpha \cap U_\beta} \text{ for each } \alpha \text{ and } \beta,$$

then there exists a unique $s \in \mathcal{F}(U)$ such that $s|_{U_\alpha} = s_\alpha$.

We made use of the following notation:

Terminology/Notation 1.3. If $s \in \mathcal{F}(V)$ is a section,and $U \subset V$ is an open subset, we define $s|_U := \mathrm{res}_{V,U}(s)$. Note that this definition is independent of the set V since the restriction maps are compatible.

Remark 1.4. When going from a presheaf to a sheaf we are requiring that we can "glue" the s_α together in a unique way. In particular, we note that if $\mathcal{F}$ is a sheaf, $U = \bigcup_\alpha U_\alpha$ is a union of open sets, and $s, t \in \mathcal{F}(U)$ such that $s|_{U_\alpha} = t|_{U_\alpha}$ for all α, then $s = t$ by the uniqueness requirement on the gluing.

Now so far we have only defined a sheaf of sets. We can now define a sheaf of rings, or R-modules, by requiring that the sections $\mathcal{F}(U)$ be rings, R-modules, etc., and by requiring the restriction maps to be ring homomorphisms (or R-module homomorphisms, etc.).

Example 1.5. The above $\mathcal{F}(U) = C(U, \mathbb{R})$ is a sheaf of $\mathbb{R}$-algebras.

Let us now give a simple example of a presheaf that is not a sheaf:

Example 1.6. Let $X = \mathbb{R}$ with the standard Euclidean topology, and for an open $U \subseteq X$, let $\mathcal{F}(U) = \{f : U \to \mathbb{R} : f(x) = c \text{ for some } c \in \mathbb{R}, x \in U\} \simeq \mathbb{R}$, and let the restriction maps be the ordinary restriction of functions. It is an exercise for the reader to show that $\mathcal{F}$ so defined is indeed a presheaf.[1] However, this *not* a sheaf in the sense that we cannot "glue" functions together; specifically, if we let $U = (0,1) \cup (2,3)$, and let $s_1 \in \Gamma((0,1), \mathcal{F})$ where $s_1 = 1$ and $s_2 \in \Gamma((2,3), \mathcal{F})$ where $s_2 = 2$. Then there does not exist an $s \in \Gamma(U, \mathcal{F})$ whose restriction to $(0,1)$ gives us s_1 and whose restriction to $(2,3)$ gives us s_2.

The reason that this example of a presheaf fails to be a sheaf is essentially that being constant on *all* of $\mathbb{R}$ is not a local property. If, however, we change the definition and consider $\mathcal{F}(U) = \{f : U \to \mathbb{R} : f \text{ is locally constant}\}$ (by locally constant we mean that for any point in U we can find a neighborhood of that point in U where f is constant in that neighborhood), then we will be able to "glue" the functions and this will give rise to a sheaf.

Now that we have the notion of a sheaf, let us define morphisms between them.

Definition 1.7. Suppose $\mathcal{F}, \mathcal{G}$ are presheaves (sheaves) on the same space X. A *morphism* $\varphi : \mathcal{F} \to \mathcal{G}$ consists of maps

$$\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U) \quad \text{for all open } U \subseteq X$$

for each open set $U \subset X$ such that these maps are compatible with restrictions, that is, if $U \subseteq V$ are open, then the following diagram commutes:

$$\begin{tikzcd}
\mathcal{F}(V) \arrow[r, "\varphi_V"] \arrow[d, "\mathrm{res}_{V,U}"'] & \mathcal{G}(V) \arrow[d, "\mathrm{res}_{V,U}"] \\
\mathcal{F}(U) \arrow[r, "\varphi_U"] & \mathcal{G}(U)
\end{tikzcd}$$

(and the restriction maps on each side is that belonging to the proper (pre)sheaf).

Example 1.8. Let $X = \mathbb{R}$ and $\mathcal{C}$ be the sheaf of continuous functions on $\mathbb{R}$. We can then define the morphism

$$\exp : \mathcal{C} \to \mathcal{C}$$

by

$$\begin{aligned} C(U, \mathbb{R}) &\to C(U, \mathbb{R}) \\ f &\mapsto \exp(f) \end{aligned}$$

for U open in X.

We are now ready to define the *stalk* of a sheaf:

Definition 1.9. If $\mathcal{F}$ is a sheaf on X and $x \in X$, is a point, then

$$\mathcal{F}_x := \varinjlim_{x \in U} \mathcal{F}(U)$$

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[1] There is an issue here in how we logically treat the restriction to the empty set. Here we set $\mathcal{F}(\emptyset) = 0$.

is the *stalk* of $\mathcal{F}$ at x (if this notation is unfamiliar to you, don't worry, we'll work out the precise definition below).

Let us define the stalk a little more explicitly. Define an equivalence relation on the set of pairs (U, s), where U is an open neighborhood of our point x and $s \in \Gamma(U, \mathcal{F}) = \mathcal{F}(U)$, where $(U, s) \sim (V, t) \iff \exists x \in W \subseteq U \cap V$ such that $s|_W = t_W$. We leave it as an exercise for the reader to check that this is indeed an equivalence relation (you'll see that what's required to make this an equivalence relation is precisely what we stipulate in the sheaf axioms). Then $\mathcal{F}_x$ is defined as the set of equivalence classes $\langle U, S\rangle$ (where we use the notation $\langle\cdot\rangle$ to denote the equivalence class of the pair (U, s)).

Example 1.10. Let $X = \mathbb{R}$, $\mathcal{F}(U) = C^\infty(U, \mathbb{R})$. For $x \in X$ then $\mathcal{F}_x$ is the space of germs of smooth functions on U at x. Note that when using germs at a point x, it makes sense to talk about the value of a germ at x and the value of its derivatives at x, but we can't say anything about the value of a germ at a point $y \neq x$.

Note that the stalk contains all of the "local" information of $\mathcal{F}$ at x.

Remark 1.11. If $\mathcal{F}$ is a sheaf of rings on X, then the stalk $\mathcal{F}_x$ has a ring structure as well defined as follows:

$$\langle U, S\rangle + \langle V, t\rangle \overset{\text{def}}{=} \langle U \cap V, s|_{U\cap V} + t|_{U\cap V}\rangle$$

and

$$\langle U, S\rangle \cdot \langle V, t\rangle \overset{\text{def}}{=} \langle U \cap V, s|_{U\cap V} \cdot t|_{U\cap V}\rangle$$

The advantage of defining all of this machinery is that once we define the structural sheaf, it will capture all of the information of the algebraic set and we'll cut our algebraic set loose from its ambient affine space.

2. The Structural Sheaf

At this point, you should forget about the Euclidean topology and recall that we are using the Zariski topology. Let $X \subseteq \mathbb{A}^n$ be an irreducible algebraic set. Recall that $A(X)$ is the affine coordinate ring (which is a domain, because X is irreducible). We define the *field of rational functions on X* as the quotient field of $A(X)$:

$$K(X) = Q(A(X)) = \{f/g : f, g \in A(X) \text{ and } g \neq 0\}$$

Recall that if $p \in X$, we have the evaluation map:

$$\begin{aligned} \mathrm{ev}_p : A(X) &\to k \\ f &\mapsto f(p) \end{aligned}$$

and recall that $\ker(\mathrm{ev}_p) = m_p = \{f \in A(X) : f(p) = 0\}$, which is a maximal ideal in $A(X)$ since the quotient $A(X)/m_p \cong k$ is a field.

Definition 2.1. The *local ring* of X at p is defined as

$$\begin{aligned}\mathcal{O}_{X,p} &= A(X)_{m_p}\\ &= \left\{\frac{f}{g} : f, g \in A(X) \text{ and } g(p) \neq 0\right\} \subset K(X)\end{aligned}$$

In other words, the local ring of X at p is the ring of rational functions on X that are defined at p.

Definition 2.2. If $U \subset X$ is an open set, then we define the *ring of regular functions on U* by

$$\mathcal{O}_X(U) = \Gamma(U, \mathcal{O}_X) \overset{\text{def}}{=} \bigcap_{p \in U} \mathcal{O}_{X,p} \subset K(X)$$

In other words, $s \in \mathcal{O}_X(U)$ if and only if for all $p \in U$ there exist $f, g \in A(X)$ such that $s = f/g$ (in particular $g(p) \neq 0$).Note that this representation of s *depends* on the point. Regular functions are not just quotients of polynomials.

Thus defined, $U \mapsto \mathcal{O}_X(U)$ is a sheaf and this call the *sheaf of regular functions* on X.