1 1/21 - Brian Katz

Throughout these notes we will let $R = k[x_1, \ldots, x_n]$ whenever there is no possibility of confusion about n. Last time we defined the Zariski topology on $\mathbb{A}^n$. Also, for every ideal $\mathfrak{a} \subset R$, we defined the algebraic set, $Z(\mathfrak{a}) = \{p \in \mathbb{A}^n : f(p) = 0 \ \forall f \in \mathfrak{a}\}$. We will spend today's class trying to describe an inverse to the function Z.

Definition 1. *Let $X \subseteq \mathbb{A}^n$, then define $I(X) = \{f \in R : f\mid_X = 0\}$. Note that this function does take algebraic sets to ideals.*

Theorem 1.1. *1) If $X \subseteq Y \subseteq \mathbb{A}^n$, then $I(Y) \subseteq I(X)$.*

2) $I(X) \cap I(Y) = I(X \cup Y)$

3) If $X \subseteq \mathbb{A}^n$ is algebraic, then $X = Z(I(X))$.

Proof. The first two claims are tautological. Note that the converse of the third claim is also trivial. Now suppose that X is an algebraic set, so $X = Z(\mathfrak{a})$ for some ideal $\mathfrak{a} \in R$. Then by definition $Z(I(X)) = \{p \in \mathbb{A}^n : f(p) = 0 \text{ for all } f \text{ such that} f\mid_X = 0\}$. Clearly every point in X satisfies this condition, so $X \subseteq Z(I(X))$. To prove the reverse inclusion, we write $Z(I(X)) = Z(I(Z(\mathfrak{a})))$. Furthermore $I(Z(\mathfrak{a})) = \{f : f\mid_{Z(\mathfrak{a})} = 0\}$ and clearly, $\mathfrak{a} \subseteq I(Z(\mathfrak{a}))$, so applying the inclusion-reversing map, Z, to both sides, we see that $Z(I(X)) = Z(I(Z(\mathfrak{a}))) \subseteq Z(\mathfrak{a}) = X$. Combining these inclusions, we have proven the final claim. ∎

Now we have defined two maps, I and Z, between the following sets:

$$\{algebraic\ sets\ in\ \mathbb{A}^n\} \underset{Z}{\overset{I}{\rightleftharpoons}} \{ideals\ of\ R\}.$$

The previous theorem proves that these two maps, when composed in one order are inverses. The next logical question is whether they are actually inverses. The answer is no. For example, in $\mathbb{R}[x]$, $Z(x^2)$ is just the point 0, and $I(0) = (x)$. However, you may note that (x) is the radical of (x^2), which we will denote $(x) = \sqrt{(x^2)}$.

Let $\mathfrak{a}$ be an ideal of R. Since R is Noetherian, $\mathfrak{a}$ is finitely generated, so say $\mathfrak{a} = (f_1, \ldots, f_s)$. Then

$$I(Z(\mathfrak{a})) = \{f \in R : f\mid_{Z(\mathfrak{a})} = 0\} = \{f \in R : \text{ if } f_1(p) = \cdots = f_s(p) = 0 \text{ then } f(p) = 0\}.$$

In particular, $a \subseteq I(Z(\mathfrak{a}))$, as we would like.

Theorem 1.2 (Hilbert Nullstellensatz). $I(Z(\mathfrak{a})) = \sqrt{(\mathfrak{a})}$

This is the same as saying: if $f \in k[x_1, \ldots, x_n]$ is such that $f_1(p) = \cdots = f_s(p) = 0$ implies $f(p) = 0$, then there exist polynomials $h_i \in R$ and a positive integer l such that $f^l = \sum_{i=1}^s h_i f_i$.

Theorem 1.3 (Weak Nullstellensatz). *If $k = \overline{k}$, then the maximal ideals of $k[x_1, \ldots, x_n]$ are of the form $M = (x_1 - a_1, \ldots, x_n - a_n)$ for $a_i \in k$.*

Remark 1.4. *1) An ideal $P \subset R$ is prime if $ab \in P$ and $a \notin P$ implies $b \in P$. Equivalently, P is prime if R/P is an integral domain.*

2) Similarly, an ideal, M, is maximal if there are no proper ideals containing it or if R/M is a field.

3) One might ask if all of the ideals of the form in the theorem are maximal. We can see that

$$k[x_1, \ldots, x_n]/(x_1 - a_1, \ldots, x_n - a_n) \cong k$$

via the isomorphism of evaluation at the point $(a_1, \ldots, a_n)$, and the ideals are maximal by the previous item.

4) The theorem is false if k is not algebraically closed. For example, by definition $\mathbb{C} = \mathbb{R}[x]/(x^2 + 1)$, but $x^2 + 1$ is not of the form described in the theorem.

proof (of Full Null., assuming Weak Null.) Let $\mathfrak{a} = (f_1, \cdots, f_s) \subset k[x_1, \ldots, x_n]$, and let f be such that $f_1(p) = \cdots = f_s(p) = 0$ implies that $f(p) = 0$. Consider the following ideal, $J \subset k[x_1, \ldots, x_{n+1}]$:

$$J := \mathfrak{a}k[x_1, \ldots, x_{n+1}] + (1 - f x_{n+1}).$$

Suppose that J is a proper ideal. Then it is contained in a maximal ideal, which by our assumption is of the form $J \subset M = (x_1 - a_1, \ldots, x_{n+1} - a_{n+1})$ for some $a_1, \ldots, a_{n+1} \in k$.

Then $f_1(a_1, \ldots, a_n) = \cdots = f_s(a_1, \ldots, a_n) = 0$, so $f(a_1, \ldots, a_n) = 0$. But any polynomial in M is zero at the point $(a_1, \ldots, a_{n+1})$, so $0 = 0 + (1 - f(a_1, \ldots, a_n)a_{n+1}) = 1 - 0 = 1$, a contradiction. Hence J is not proper. Thus $J = k[x_1, \ldots, x_{n+1}]$, so $1 \in J$. Hence

$$1 = \sum_{i=1}^{s} h_i(x_1, \ldots, x_{n+1}) f_i(x_1, \ldots, x_n) + h_{s+1}(x_1, \ldots, x_{n+1})(1 - f(x_1, \ldots, x_n)x_{n+1}).$$

Set $x_{n+1} = \frac{1}{f}$ to get the equality

$$1 = \sum_{i=1}^{s} h_i(x_1, \ldots, x_n, 1/f) f_i(x_1, \ldots, x_n),$$

which in turn implies (by clearing the denominators), that

$$f^l = \sum_{i=1}^{s} \tilde{h}_i(x_1, \ldots, x_n) f_i(x_1, \ldots, x_n).$$

□

Theorem 1.5. *There is a bijection between algebraic sets in $\mathbb{A}^n$ and radical ideals in R.*

To an algebraic set, X we associate its ideal $I(X)$, and to a radical ideal $\mathfrak{a}$ we associate its zero locus $Z(\mathfrak{a})$. As we have proven above, $Z(I(X)) = X$, and $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}} = \mathfrak{a}$.

Example 1. *Let X be the union of a line and a plane in $\mathbb{A}^3$, for example $L : x = y = 0$ and $H : z = 0$. Then $I(X) = I(L \cup H) = I(L) \cap I(H) = (x, y) \cap (z) = (xz, yz)$. This last equality is actually an unpleasant calculation that we will include this once.*

Suppose $ax + by = cz$; we wish to show that z divides both a and b, or at least that it can be manipulated to be in that form. Write $a = zf + a_1(x, y)$ and $b = zg + b_1(x, y)$. Then $z(fx + gy) + (a_1x + b_1y) = cz$. Because the second term contains no z but is divisible by z, it must be zero. Hence we can rewrite our original element as $fxz + gyz$.

Example 2. *1. Consider the two points, $(0,1)$ and $(1,0)$ in $\mathbb{A}^2$. The ideal of the first is $I_1 = (x, y-1)$ and of the second is $I_2 = (x-1, y)$. Then*

$$I_1 \cap I_2 = \sqrt{I_1 \cap I_2} = \sqrt{I_1 I_2} = (x^2 - x, xy, (x-1)(y-1), y^2 - y).$$

2. Let $X = Z(I)$ and $Y = Z(J)$. Then $X \cap Y = Z(I) \cap Z(J) = Z(I \cup J) = Z(\sqrt{I+J})$

3. Consider the intersection of the two curves $y = x^2 - 1$ and $y = 0$ in $\mathbb{A}^2$. These varieties have ideals $I = (y)$ and $J = (y - x^2 + 1)$. Then $\sqrt{I+J} = (y, x^2 - 1)$.

4. Similarly, If $I = (y)$ and $J = (y - x^2)$, then $\sqrt{I+J} = \sqrt{(y, x^2)} = (x, y)$. The exponent in the penultimate term corresponds to the fact that the parabola hits the line with multiplicity 2. This is why we have to take the radical.

Definition 2. *A topological space X is said to be noetherian if it satisfies the descending chain condition on closed sets: for every chain $Y_1 \supseteq Y_2 \supseteq \cdots$ with $Y_i \subseteq X$ there exists an r such that $Y_r = Y_{r+1} = \cdots$.*

Example 3. *$\mathbb{A}^n$ is a noetherian space.*

Proposition 1.6. *If X is noetherian then every closed $\emptyset \neq Y \subseteq X$ can be expressed as $Y = Y_1 \cup \cdots \cup Y_r$ where Y_i are irreducible closed subsets of X such that $Y_i \not\subseteq Y_j$ for $i \neq j$. Such a decomposition is unique (up to the order) and the Y_i are called the irreducible components of Y.*

Proof. Note that X is noetherian implies that every non-empty collection, S, of closed subsets of X has a minimal element. Suppose $S \neq \emptyset$ is the collection of closed subsets of X which cannot be written as in the statement and let Y be a minimal element of S. Note that this means that Y is reducible. Then there exist proper, closed Y_1 and Y_2 in Y such that $Y = Y_1 \cup Y_2$. By the minimality of Y, $Y_1, Y_2 \notin S$. Hence $Y_1 = Y_1' \cup \cdots \cup Y_s'$ and $Y_2 = Y_1'' \cup \cdots \cup Y_t''$, where the Y_i' and Y_j'' are closed and irreducible. Then $Y = (\cup Y_i') \cup (\cup Y_j'')$. By throwing away of these sets, we get $Y = \cup Y_k$ with Y_k closed and irreducible and $Y_k \subsetneq Y_{k'}$ if $k \neq k'$. This contradicts our assumption that S is non-empty, and the theorem is proved. □

2 Morphisms - 1/26 - Miner

Let $X \subseteq \mathbb{A}^n$ be an irreducible algebraic set. Define $\mathcal{A}(X) = R/I(X)$, which we will call the affine coordinate ring of X. So what should be morphisms between irreducible algebraic sets?

Example 4. *The map $(x, y) \mapsto x$ from the curve $y = x^2$ in $\mathbb{A}^2$ to $\mathbb{A}^1$ should certainly be one. So we can make the following, tentative definition.*

Definition 3. *Let $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$ be an irreducible algebraic sets. A morphism $\alpha : X \to Y$ is given by polynomials $f_i \in k[x_1, \ldots, x_n]$ such that*

$$\alpha(x_1, \ldots, x_n) = (f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n)).$$

Then α induces a morphism $\alpha^* : \mathcal{A}(Y) = k[y_1, \ldots, y_m]/I(Y) \to \mathcal{A}(X) = k[x_1, \ldots, x_n]/I(X)$ between the affine coordinate rings of Y and X as follows. Let $\phi(y_1, \ldots, y_m) \in \mathcal{A}(Y)$. Then

$$\alpha^*(\phi(y_1, \ldots, y_m)) = \phi(f_1, \ldots, f_m)$$

where $\phi(f_1, \ldots, f_m)$ is seen to be a polynomial in $x_1, \ldots, x_n$.

Certainly, α^* makes sense as a map $k[y_1, \ldots, y_m] \to \mathcal{A}(X)$. Suppose $\phi \in I(Y)$. Then $\phi(f_1, \ldots, f_m) \mid_X = 0$, so $\alpha^*(\phi) \in I(X)$. Then α^* is a morphism of k-algebras and is determined by α. We will soon prove that α^* determines α, which says that thinking about α or about α^* in the end amounts to the same thing. This shows that the affine coordinate ring of an affine variety (which is an algebraic object), completely determines the geometry of the variety. To show this, we look at how α^* acts on each y_i:

$$\alpha^*(y_i) = f_i \bmod I(X)$$

(determined up to the ideal $I(X)$) gives

$$\alpha^*(y_i) : Y \to k$$

and $f_i : X \to k$ is well-defined, so that we recover α:

$$\alpha(x) = (\alpha^*(y_1), \ldots, \alpha^*(y_m))(x) \ \in \mathbb{A}^m.$$

Claim: $\alpha(X) \subseteq Y$.

We need to show that $\alpha(X)$ is annihilated by every element of $I(Y)$. Pick $g \in I(Y)$ We need to show that $g(\alpha(x)) = 0$. This is equivalent with showing that

$$g(\alpha^*(y_1), \ldots, \alpha^*(y_m))(x) = 0 \Leftrightarrow \alpha^*(g(y_1, \ldots, y_m)) = 0$$

(α^* is a ring homomorphism). But $\alpha^*(I(Y)) \subseteq I(X)$, which means

$$\alpha(g(y_1, \ldots, y_m)) \in I(X) \Leftrightarrow \alpha^*(g(y_1, \ldots, y_m))|_X = 0.$$

Thus $\alpha : X \to Y$ is determined by α^*.

We have proved:

Theorem 2.1. *There is a 1-1 correspondence between homomorphisms of algebraic sets and homomorphisms of k-algebras:*

$$\mathrm{Hom}_{\text{alg. sets}}(X, Y) \longleftrightarrow \mathrm{Hom}_{k-\text{alg}}(\mathcal{A}(Y), \mathcal{A}(X))$$

where

$$(\alpha : X \to Y) \longleftrightarrow (\alpha^* : \mathcal{A}(Y) \to \mathcal{A}(X))$$

So, we have a way to tell when two algebraic sets are isomorphic: check if their corresponding affine coordinate rings are isomorphic.

Example 5. *Let $Y = \mathbb{A}^1$ so that $\mathcal{A}(Y) = k[t]$. What is Hom$(X, \mathbb{A}^1)$?*

$$\{f : X \to k\} = \mathrm{Hom}(X, \mathbb{A}^1) = \mathrm{Hom}_{k-\text{alg}}(k[t], \mathcal{A}(X)) \cong \mathcal{A}(X)$$

$\mathrm{Hom}_{k-\text{alg}}(k[t], \mathcal{A}(X))$ *is determined uniquely by where t goes:*

$$t \rightsquigarrow u \in \mathcal{A}(X). \textit{ (Affine coordinate ring of } X) \longleftrightarrow (\textit{ morphisms from } X \textit{ to } \mathbb{A}^1).$$

Now recall that we have defined morphisms between two algebraic sets and we prove that they are continuous in the Zariski topology:

Proposition 2.2. *Let $X \subseteq \mathbb{A}^n$, $Y \subseteq \mathbb{A}^m$, and let $\alpha : X \to Y$ be a morphism. Then α is continuous in the Zariski topology.*

Proof. First, $\alpha = (f_1, \ldots, f_m)$. We need to show that if $Z \subseteq Y$ is closed, then $\alpha^{-1}(Z)$ is also closed. Now, $I(Z) =< h_1, \ldots, h_s >$, where $h_i \in \mathcal{A}(Y)$. So,

$$\begin{aligned}\alpha^{-1}(Z) &= \{x \in X : h_1(\alpha(x)) = \ldots = h_s(\alpha(x)) = 0\} \\ &= \{x \in X : h_1(f_1(x), \ldots, f_m(x)) = 0, \ldots, h_s(f_1(x), \ldots, f_m(x)) = 0\}\end{aligned}$$

And $\alpha^{-1}(Z)$ is thus seen to be Zariski closed in X. Therefore, α is continuous. □

Example 6. *Let L be the line $5x + 3y = 2 \subset \mathbb{A}^2$. It should be that $L \cong \mathbb{A}^1$. To prove this, we will show their affine coordinate rings are isomorphic: Now,*

$$\mathcal{A}(L) \cong \mathcal{A}(\mathbb{A}^1),$$

is true since

$$\frac{k[x, y]}{(5x + 3y - 2)} \cong k[t],$$

so it follows that $L \cong \mathbb{A}^1$.

Example 7. *$X : (y^2 = x^3) \subset \mathbb{A}^2$. In you draw this curve over $\mathbb{R}$, you see that it has two branches meeting at the point $(0, 0)$ which is a cusp. We define the morphism $f : \mathbb{A}^1 \to X$, by $f(t) = (t^2, t^3)$. f is seen to be bijective, hence a homeomorphism in the Zariski topology. Now, we claim that f is not an isomorphism:*

$$f^* : \mathcal{A}(X) \to k[t] \text{ with } \mathcal{A}(X) = \frac{k[x, y]}{(x^2 - y^3)}$$

and since $f^*(\phi(x, y)) = \phi(t^2, t^3)$ it is clear that f^* cannot be surjective, as $t \notin Im(f^*)$. So, $X \ncong \mathbb{A}^1$, and f is an example of a homeomorphism which is not an isomorphism.

Example 8. *(**<u>Frobenius</u>**) Let $k = \bar{k}$ and $char(k) = p > 0$. Consider $F : \mathbb{A}^1 \to \mathbb{A}^1$ defined by $F(t) = t^p$. Then $F^* : k[t] \to k[t]$ is the induced morphism that sends*

$$(a_0 + a_1 t + \ldots + a_s t^s) \rightsquigarrow (a_0 + a_1 t^p + \ldots + a_s t^{ps}).$$

F is another example of a homeomorphism that is not an isomorphism (the fixed points of F are the points in $\mathbb{F}_p$).

<u>Moral:</u> Morphisms of algebraic sets are more than just continuous functions. (This info will be contained in sheafs.) So, we will need more than the Zariski topology to describe morphisms.

Definition 4. *(**Localization**) Let R be a ring, and let $S \subset R$ be a multiplicative set, $(0 \notin S)$.*

$$S^{-1}R := \{\frac{a}{b} : a \in R,\ b \in S\}$$

Define an equivalence relation $\sim$ on this set by:

$$(a, b) \sim (c, d) \Leftrightarrow \exists s \in S \text{ such that } s(ad - bc) = 0.$$

Remark 2.3. *If R is a domain, and $\mathfrak{p}$ is a prime ideal, let $S = R - \mathfrak{p}$. Then*

$$\begin{aligned} S^{-1}R &= \{\tfrac{x}{y} : x \in R,\ y \notin \mathfrak{p}\} \\ &= \text{localization of } R \text{ at } \mathfrak{p} = R_{\mathfrak{p}}. \end{aligned}$$

In particular, $R_{(0)} = Q(R) =$ the fraction field of R

(Sheaves are very widespread in math. Their definition makes sense in any topological space.)

Definition 5. *Let X be a topological space. A presheaf $\mathcal{F}$ on X consists of the following data:*

i. $\forall(\text{open})U \subseteq X$ we associate a set $\mathcal{F}(U) = \Gamma(U, \mathcal{F})$ (called sections of $\mathcal{F}$ over U)

ii. For open sets $U \subseteq V \subseteq X$(we associate restriction maps) $\text{res}_{V,U} : \mathcal{F}(V) \to \mathcal{F}(U)$

satisfying the following axioms:
1. *$res_{U,U} : \mathcal{F}(U) \to \mathcal{F}(U)$ is the identity map $1_{\mathcal{F}(U)}$*
2. *Compatibility: If $U \subseteq V \subseteq W$, then the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{F}(W) & \xrightarrow{\phi} & \mathcal{F}(V) \\ & {\scriptstyle\psi}\searrow & \downarrow{\scriptstyle\theta} \\ & & \mathcal{F}(U) \end{array}$$

where $\phi = res_{W,V}$, $\psi = res_{W,U}$, $\theta = res_{V,U}$. In other words, $res_{W,U} = res_{V,U} \circ res_{W,V}$

Example 9. *Let $X = \mathbb{R}^n$ with the Euclidean topology, $\mathcal{F}$ be a presheaf of continuous functions. So for each open set $U \subset \mathbb{R}^n$, we have that*

$$\mathcal{F}(U) = \mathcal{C}(U, \mathbb{R}) = \{f : U \to \mathbb{R},\ \textit{such that } f \textit{ is continuous}\}.$$

If $U \subseteq V \subseteq \mathbb{R}^n$ then $\mathcal{F}(V) \to \mathcal{F}(U)$ (that is, $(f : V \to \mathbb{R}) \to (f|_U : U \to \mathbb{R})$) is defined to be the ordinary restriction map. It is left as an exercise to check that the axioms are satisfied.

Example 10. *Let $X = \mathbb{R}^n$, and let $\mathcal{G}$ be the pre-sheaf of $\mathcal{C}^\infty$-differentiable function. Then $\mathcal{G}(U) = \{f : U \to \mathbb{R} :\ f \text{ is } \mathcal{C}^\infty\}$*

Example 11. *Let $X = \mathbb{C}$. Then the following is a presheaf:*

$$U \rightsquigarrow \mathcal{T}_{hol}(U) = \{f : U \to \mathbb{C} :\ f \textit{ is holomorphic}\}.$$

3 Sheaves Cont. - 1/28 - Fili

Last time we defined a *presheaf* on a topological space X which assigned to each open set $U \subseteq X$ a space of sections $\mathcal{F}(U) = \Gamma(U, \mathcal{F})$, along with restriction maps for open subsets $V \subseteq U \subset X$,

$$\text{res}_{V,U} : \mathcal{F}(U) \to \mathcal{F}(V)$$

satisfying the usual compatibility condition. Let us now consider an example.

Example 12. *Let $X = \mathbb{R}^n$ with the euclidean topology and define*

$$\mathcal{F}(U) = C(U, \mathbb{R}) := \{f : U \to \mathbb{R} \ : \ f \text{ is continuous}\},$$

which is the (pre)sheaf of continuous real-valued functions on X.

We can now define a *sheaf*:

Definition 6. *A presheaf $\mathcal{F}$ on X is a* sheaf *if the following gluing condition is satisfied: if $U \subseteq X$ is an open set and $\{U_\alpha\}_\alpha$ an open cover of U and $\{s_\alpha \in \mathcal{F}(U_\alpha)\}_\alpha$ a collection of elements such that*

$$s_\alpha|_{U_\alpha \cap U_\beta} = s_\beta|_{U_\alpha \cap U_\beta} \text{ for each } \alpha \text{ and } \beta,$$

then there exists a unique $s \in \mathcal{F}(U)$ such that $s|_{U_\alpha} = s_\alpha$.

We made use of the following notation:

Definition 7. *If $s \in \mathcal{F}(V)$ is a section and $U \subset V$ is an open subset, we define $s|_U := \mathrm{res}_{V,U}(s)$. Note that this definition is independent of the set V since the restriction maps are compatible.*

Remark 3.1. *When going from a presheaf to a sheaf we are requiring that we can "glue" the s_α together in a unique way. In particular, we note that if $\mathcal{F}$ is a sheaf, $U = \bigcup_\alpha U_\alpha$ is a union of open sets, and $s, t \in \mathcal{F}(U)$ such that $s|_{U_\alpha} = t|_{U_\alpha}$ for all α, then $s = t$ by the uniqueness requirement on the gluing.*

Now so far we have only defined a sheaf of sets. We can now define a sheaf of rings, or R-modules, by requiring that the sections $\mathcal{F}(U)$ be rings, R-modules, etc., and by requiring the restriction maps to be ring homomorphisms (or R-module homomorphisms, etc.).

Example 13. *The above $\mathcal{F}(U) = C(U, \mathbb{R})$ is a sheaf of $\mathbb{R}$-algebras.*

Let us now give a simple example of a presheaf that is not a sheaf:

Example 14. *Let $X = \mathbb{R}$ with the standard Euclidean topology, and for an open $U \subseteq X$, let $\mathcal{F}(U) = \{f : U \to \mathbb{R} : f(x) = c \text{ for some } c \in \mathbb{R}, x \in U\} \simeq \mathbb{R}$, and let the restriction maps be the ordinary restriction of functions. It is an exercise for the reader to show that $\mathcal{F}$ so defined is indeed a presheaf.* [1] *However, this* not *a sheaf in the sense that we cannot "glue" functions together; specifically, if we let $U = (0,1) \cup (2,3)$, and let $s_1 \in \Gamma((0,1), \mathcal{F})$ where $s_1 = 1$ and $s_2 \in \Gamma((2,3), \mathcal{F})$ where $s_2 = 2$. Then there does not exist an $s \in \Gamma(U, \mathcal{F})$ whose restriction to $(0,1)$ gives us s_1 and whose restriction to $(2,3)$ gives us s_2.*

The reason that this example of a presheaf fails to be a sheaf is essentially that being constant on *all* of $\mathbb{R}$ is not a local property. If, however, we change the definition and consider $\mathcal{F}(U) = \{f : U \to \mathbb{R} : f \text{ is locally constant}\}$ (by locally constant we mean that for any point in U we can find a neighborhood of that point in U where f is constant in that neighborhood), then we will be able to "glue" the functions and this will give rise to a sheaf.

Now that we have the notion of a sheaf, let us define morphisms between them.

[1] There is an issue here in how we logically treat the restriction to the empty set. Here we set $\mathcal{F}(\emptyset) = 0$.

Definition 8. *Suppose $\mathcal{F}, \mathcal{G}$ are presheaves (sheaves) on the same space X. A* morphism *$\varphi : \mathcal{F} \to \mathcal{G}$ consists of maps*

$$\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U) \quad \textit{for all open } U \subseteq X$$

for each open set $U \subset X$ such that these maps are compatible with restrictions, that is, if $U \subseteq V$ are open, then the following diagram commutes:

$$\begin{array}{ccc} \mathcal{F}(V) & \xrightarrow{\varphi_V} & \mathcal{G}(V) \\ {\scriptstyle \mathrm{res}_{V,U}}\downarrow & & \downarrow{\scriptstyle \mathrm{res}_{V,U}} \\ \mathcal{F}(U) & \xrightarrow{\varphi_U} & \mathcal{G}(U) \end{array}$$

(and the restriction maps on each side is that belonging to the proper (pre)sheaf).

Example 15. *Let $X = \mathbb{R}$ and $\mathcal{C}$ be the sheaf of continuous functions on $\mathbb{R}$. We can then define the morphism*

$$\exp : \mathcal{C} \to \mathcal{C}$$

by

$$\begin{aligned} C(U, \mathbb{R}) &\to C(U, \mathbb{R}) \\ f &\mapsto \exp(f) \end{aligned}$$

for U open in X.

We are now ready to define the *stalk* of a sheaf:

Definition 9. *If $\mathcal{F}$ is a sheaf on X and $x \in X$, is a point, then*

$$\mathcal{F}_x := \varinjlim_{x \in U} \mathcal{F}(U)$$

is the stalk *of $\mathcal{F}$ at x (if this notation is unfamiliar to you, don't worry, we'll work out the precise definition below).*

Let us define the stalk a little more explicitly. Define an equivalence relation on the set of pairs (U, s), where U is an open neighborhood of our point x and $s \in \Gamma(U, \mathcal{F}) = \mathcal{F}(U)$, where $(U, s) \sim (V, t) \iff$ there exists a neighborhood $x \in W \subseteq U \cap V$ such that $s|_W = t|_W$. We leave it as an exercise for the reader to check that this is indeed an equivalence relation (you'll see that what's required to make this an equivalence relation is precisely what we stipulate in the sheaf axioms). Then $\mathcal{F}_x$ is defined as the set of equivalence classes $\langle U, s \rangle$ (where we use the notation $\langle \cdot \rangle$ to denote the equivalence class of the pair (U, s)).

Example 16. *Let $X = \mathbb{R}$, $\mathcal{F}(U) = C^\infty(U, \mathbb{R})$. For $x \in X$ then $\mathcal{F}_x$ is the space of germs of smooth functions on U at x. Note that when using germs at a point x, it makes sense to talk about the value of a germ at x and the value of its derivatives at x, but we can't say anything about the value of a germ at a point $y \neq x$.*

Note that the stalk contains all of the "local" information of $\mathcal{F}$ at x.

Remark 3.2. *If $\mathcal{F}$ is a sheaf of rings on X, then the stalk $\mathcal{F}_x$ has a ring structure as well defined as follows:*

$$\langle U, S\rangle + \langle V, t\rangle \stackrel{\text{def}}{=} \langle U \cap V, s|_{U\cap V} + t|_{U\cap V}\rangle$$

and

$$\langle U, S\rangle \cdot \langle V, t\rangle \stackrel{\text{def}}{=} \langle U \cap V, s|_{U\cap V} \cdot t|_{U\cap V}\rangle$$

The advantage of defining all of this machinery is that once we define the structural sheaf, it will capture all of the information of the algebraic set and we'll cut our algebraic set loose from its ambient affine space.

4 The Structural Sheaf

At this point, you should forget about the Euclidean topology and recall that we are using the Zariski topology. Let $X \subseteq \mathbb{A}^n$ be an irreducible algebraic set. Recall that $A(X)$ is the affine coordinate ring (which is a domain, because X is irreducible). We define the *field of rational functions on X* as the quotient field of $A(X)$:

$$K(X) = Q(A(X)) = \{f/g : f, g \in A(X) \text{ and } g \neq 0\}$$

Recall that if $p \in X$, we have the evaluation map:

$$\begin{aligned} \text{ev}_p : A(X) &\to k \\ f &\mapsto f(p) \end{aligned}$$

and recall that $\ker(\text{ev}_p) = m_p = \{f \in A(X) : f(p) = 0\}$, which is a maximal ideal in $A(X)$ since the quotient $A(X)/m_p \cong k$ is a field.

Definition 10. *The* local ring *of X at p is defined as*

$$\begin{aligned} \mathcal{O}_{X,p} &= A(X)_{m_p} \\ &= \left\{ \frac{f}{g} : f, g \in A(X) \text{ and } g(p) \neq 0 \right\} \subset K(X) \end{aligned}$$

In other words, the local ring of X at p is the ring of rational functions on X that are defined at p.

Definition 11. *If $U \subset X$ is an open set, then we define the* ring of regular functions on U *by*

$$\mathcal{O}_X(U) = \Gamma(U, \mathcal{O}_X) \stackrel{\text{def}}{=} \bigcap_{p \in U} \mathcal{O}_{X,p} \subset K(X)$$

In other words, $s \in \mathcal{O}_X(U)$ if and only if for all $p \in U$ there exist $f, g \in A(X)$ such that $s = f/g$ (in particular $g(p) \neq 0$). Note that this representation of s depends *on the point. Regular functions are not just quotients of polynomials.*

Thus defined, $U \mapsto \mathcal{O}_X(U)$ is a sheaf and this call the *sheaf of regular functions* on X.

Let $X \subseteq \mathbb{A}^n$ be an irreducible algebraic set. Our goal is to think about this set X without reference to the ambient space $\mathbb{A}^n$. To this end, recall the domain $A(X)$, the affine coordinate ring of X. Then $Q(A(X)) = k(X) =$ the field of rational functions on X. That is,

$$Q(A(X)) = \left\{ \frac{f}{g} \mid f, g \text{ polynomials on } X \right\}.$$

Definition 12. *Let $p = (a_1, \ldots, a_n) \in X$, and define*

$$\mathcal{O}_{X,p} = A(X)_{\mathcal{M}_p}, \quad \mathcal{M}_p = \{f \in A(X) \ : \ f(p) = 0\}.$$

This is called a local ring at p. That is, $\mathcal{O}_{X,p}$ is a local ring with maximal ideal

$$\left\{ \frac{f}{g} \ : \ f, g \in A(X), g(p) \neq 0 \right\}$$

We would like to create a sheaf $\mathcal{O}_X$. To this end, let $U \subseteq X$ be an open set in X. Then we define a map

$$U \mapsto \mathcal{O}_X(U) = \bigcap_{p \in U} \mathcal{O}_{X,p} \subseteq k(X).$$

Note that this is well-defined. Thus we have

$$\mathcal{O}_X(U) = \left\{ s \in k(X) \ : \ \forall \ p \in U, \exists \ f, g \in A(X) \ : g(x) \neq 0, s = \frac{f}{g} \right\}.$$

In particular, think of elements in $\mathcal{O}_X(U)$ as set-theoretic functions defined on U:

$$s : U \to k \ : \ s(x) = \frac{f(x)}{g(x)}.$$

That is, functions on U expressable locally as polynomials.

Remark 4.1. *In general, for $s \in \mathcal{O}_X(U)$, it's not true that $s = \frac{f}{g}$ globally on U, $f, g \in A(X)$, as we might have $g(x) = 0$ for some values of x.*

Example 17. *Let $X \subseteq \mathbb{A}^4$ be the space $X : xw = yz$. Define*

$$X_W = \{p \in X \ : \ w \neq 0\}, \quad X_Y = \{p \in X \ : \ y \neq 0\}.$$

Let

$$U = X_W \cup X_Y.$$

Then we can define a regular function s on U by:

$$s(p) = \begin{cases} \frac{x}{y} \text{ on } X_Y \\ \frac{z}{w} \text{ on } X_W \end{cases}$$

Then $s(p) \in \mathcal{O}_X(U)$, s is algebraic on each subset, and since $\frac{x}{y} = \frac{z}{w}$ on $X_W \cap X_Y$ it agrees on the overlap. But we cannot find a single function that will work on the entire U.

Remark 4.2. *A rigorous proof of this requires the concept of dimension, which we haven't yet discussed. Thus all that we can currently say is that there is no intuitive definition of such a function.*

It turns out that the association

$$U \mapsto \mathcal{O}_X(U)$$

is a sheaf. If $U \subseteq V$ are both open subsets of X, then we define the restriction map by inclusion:

$$\mathcal{O}_X(U) \hookrightarrow \mathcal{O}_X(V)$$

as both $\mathcal{O}_X(U)$ and $\mathcal{O}_X(V)$ are contained in $k(X)$.

Question 4.3. *What is $\mathcal{O}_X(\emptyset)$?*

Our two texts have definition discrepancies. In Hartshorne, if $\mathcal{F}$ is a sheaf of rings, then $\mathcal{F}(\emptyset) = 0$ by part of the definition of a sheaf. But it is clear that we actually have $\mathcal{O}_X(\emptyset) = k(X)$.

We can understand $\mathcal{O}_X(U)$ for some distinguished open sets. We have a distinguished basis for the Zariski topology on X given by the following. Let $f \in A(X)$. Then define

$$X_f = X - Z(f) = \{p \in X \ : \ f(p) \neq 0\}.$$

The set $\{X_f\}_{f \in A(X)}$ is a basis for the topology of X.

Proposition 4.4. $\mathcal{O}_X(X_f) = A(X)_f = \left\{ \frac{g}{f^r} \ : \ r \geq 0 \right\}$

Remark 4.5. *To relate this with our previous notation for a localized ring, let $S = \{f^r \ : \ r \geq 0\}$. Then $S^{-1}R = R_f$.*

Proof. It is clear that $A(X)_f \subseteq \mathcal{O}_X(X_f)$. Thus let's consider the opposite inclusion.

Suppose that $s \in \mathcal{O}_X(X_f)$. Then define the ideal

$$B := \{h \in A(X) \ : \ hs \in A(X)\}.$$

This is an ideal which depends on s. If we show that $f^r \in B$, then we are done, since

$$\begin{aligned} f^r \in B \quad &\iff \quad f^r s \in A(X) \\ &\iff \quad s = \frac{g(x)}{f^r}, \ g(x) \in A(X) \end{aligned}$$

We know that $s \in \mathcal{O}_X(X_f)$. Pick some $x \in X_f$. Then there exist some functions $g, h \in A(X)$ such that $s = \frac{g}{h}$, $h \neq 0$, by definition. Thus we have

$$\begin{aligned} sh = g \in A(X) \quad &\implies \quad h \in B, h(x) \neq 0 \\ &\implies \quad x \notin Z(B) \\ &\implies \quad X_f \cap Z(B) = \emptyset \\ &\implies \quad \text{(By Nullstellensatz)} \sqrt{(f)} = I(Z(f)) \subseteq I(Z(B)) = \sqrt{B} \\ &\implies \quad f \in \sqrt{B} \\ &\implies \quad \exists \ r \text{ such that } f^r \in B. \end{aligned}$$

□

Remark 4.6. $\mathcal{O}_X(U \cup V) = \mathcal{O}_X(U) \cap \mathcal{O}_X(V) \subseteq k(X)$. *Any open* $U \subset X = \bigcup_{f_\alpha \in A(X)} X_{f_\alpha}$, *and* $\mathcal{O}_X(U) = \bigcap_\alpha \mathcal{O}(X_{f_\alpha}) = \bigcap_\alpha A(X)_{f_\alpha}$.

Corollary 4.7. $\mathcal{O}_X(X) = A(X)$. *That is, the only regular functions on all of* X *are polynomials (global regular functions) So the old way of thinking about morphisms is consistent with the new definition.*

Proposition 4.8. *The stalk of* $\mathcal{O}_X$ *at* $p \in X$ *is* $\mathcal{O}_{X,p} = A(X)_{\mathcal{M}_p}$.

Proof. Recall that

$$\mathcal{O}_{X,p} = \{\langle U, s\rangle \ : \ U \subseteq X \text{ an open neighborhood of } p,\ s \in \mathcal{O}_X(U)\}.$$

Define a map $\eta : \mathcal{O}_{X,p} \to A(X)_{\mathcal{M}_p}$ by

$$\langle U, s\rangle \mapsto \frac{f}{g}$$

where by assumption $s = \frac{f}{g}$, $g(p) \neq 0$ and $f, g \in A(X)$.

We will show that η is a bijection. That η is injective is fairly straightforward, so we will concentrate on showing surjectivity. Let $\frac{f}{g} \in A(X)_{\mathcal{M}_p}$. Then $\frac{f}{g}$ defines a regular function in $\mathcal{O}_X(X_g)$. Thus we have

$$\eta(\langle X_g, \frac{f}{g}\rangle) = \frac{f}{g}.$$

So η is surjective. It follows that $\mathcal{O}_{X,p} = A(X)_{\mathcal{M}_p}$. $\square$

Example 18. *Consider* $\mathcal{O}_{\mathbb{A}^2}$. *Then*

$$k(\mathbb{A}^2) = k(x,y) = \text{ field of rational functions}$$

$$A(\mathbb{A}^2) = k[x,y] = \text{ ring of regular functions}$$

Moreover, the stalk of $\mathbb{A}^2$ *at* 0 *is as follows:*

$$\mathcal{O}_{\mathbb{A}^2,0} = k[x,y]_{\mathcal{M}_0} = \left\{\frac{f(x,y)}{g(x,y)} \ : \ g(0,0) \neq 0\right\}.$$

Let $U = \mathbb{A}^2$. *Then* $\Gamma(U, \mathcal{O}_{\mathbb{A}^2}) = \{$ *regular functions on all of* $\mathbb{A}^2\} = k[x,y]$.

Example 19. *Let* $V = \mathbb{A}^2 \setminus \{x = 0\} = \mathbb{A}^2_x$. *Then recall that* $X_f = \{p \in X \ : \ f(p) \neq 0\}$. *Thus we have*

$$\begin{aligned} \Gamma(V, \mathcal{O}_{\mathbb{A}^2}) &= k[x,y]_x = \{\frac{f(x,y)}{x^a} \ : \ a \geq 0\} \\ &= k[x,y][\frac{1}{x}]. \end{aligned}$$

Example 20. *Let* $V_1 = \mathbb{A}^2 \setminus \{y^2 = x\}$. *Then* $\Gamma(V_1, \mathcal{O}_{\mathbb{A}^2}) = k[x, y, \frac{1}{y^2 - x}]$.

Example 21. *Let* $W = \mathbb{A}^2 \setminus \{(0,0\} = (\mathbb{A}^2 \setminus \{x=0\}) \cup (\mathbb{A}^2 \setminus \{y=0\}) = \mathbb{A}^2_x \cup \mathbb{A}^2_y$*. Then*

$$\Gamma(W, \mathcal{O}_{\mathbb{A}^2}) = \Gamma(\mathbb{A}^2_x, \mathcal{O}_{\mathbb{A}^2}) \cap \Gamma(\mathbb{A}^2_y, \mathcal{O}_{\mathbb{A}^2}) = k[x,y]_x \cap k[x,y]_y$$

To figure out what this intersection is, note that this is the set of functions $h(x,y) = \frac{f(x,y)}{x^a} = \frac{g(x,y)}{y^b}$ *where* $f, g \in A(\mathbb{A}^2)$*. But this implies that* $y^b f(x,y) = x^a g(x,y)$*, hence that* $x^a \mid f$*,* $y^b \mid g$ *and hence* $h(x,y) \in A(\mathbb{A}^2)$

It follows that

$$\Gamma(\mathbb{A}^2 \setminus \{(0,0)\}, \mathcal{O}_{\mathbb{A}^2}) = \Gamma(\mathbb{A}^2, \mathcal{O}_{\mathbb{A}^2}).$$

This is the analogue of Hartog's theorem in Analysis, which is the same statement except about Holomorphic functions.

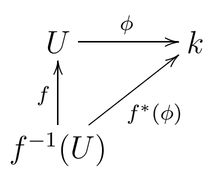
5 2/2 - Meth

Let $X \subseteq \mathbb{A}^n$ be an irreducible algebraic set and $\mathcal{O}_X$ its structure sheaf of regular functions.

Let $X \subseteq \mathbb{A}^n$, $Y \subseteq \mathbb{A}^m$ be irreducible algebraic sets, $f : X \to Y$ any continuous map. Then for any open set $U \subseteq Y$ and for any function $\phi : U \to k$, we define

$$f^*(\phi) := \phi \circ f : f^{-1}(U) \to k.$$

That is, define $f^*(\phi)$ so that the following diagram commutes:



Theorem 5.1. *Let* $X \subseteq \mathbb{A}^n$*,* $Y \subseteq \mathbb{A}^m$ *be irreducible algebraic sets and* $f : X \to Y$ *a continuous map. Then the following are equivalent:*

1 f *is a morphism.*

2 $\forall U \subseteq Y$ *open,*

$$\begin{array}{rcl} f^* : \mathcal{O}_Y(U) & \to & \mathcal{O}_X(f^{-1}(U)) \\ \phi & \mapsto & \phi \circ f \end{array}$$

is a ring homomorphism (that is, f^* *maps regular functions to regular functions).*

3

$$\begin{array}{rcl} f^* : \mathcal{O}_Y(Y) & \to & \mathcal{O}_X(X) \\ || & & || \\ A(Y) & & A(X) \end{array}$$

is a ring homomorphism.

Proof. 1 ⇒ 2: Suppose $f : X \to Y$ is a morphism, so $f = (f_1, \ldots, f_m)$, $f_i \in k[x_1, \ldots, x_n]$. Let $\phi : U \to k$, $\phi \in \mathcal{O}_Y(U)$, so $f^*\phi = \phi \circ f : f^{-1}(U) \to k$. Pick $x \in f^{-1}(U)$, then $f(x) \in U$ so $\exists g, h \in A(Y)$, $\phi = g/h$, and $h(f(x)) \neq 0$. Now $f^*(\phi) = \phi(f_1, \ldots, f_m) = \frac{g(f_1,\ldots, f_m)}{h(f_1,\ldots, f_m)}$, where $h(f_1, \ldots, f_m)(x) = h(f(x)) \neq 0$, so $\frac{g(f_1,\ldots, f_m)}{h(f_1,\ldots, f_m)} \in \mathcal{O}_X(f^{-1}(U))$. The fact that it is a ring homomorphism is immediate.

2 ⇒ 3: Obvious if we set $U = Y$.

3 ⇒ 1: Suppose $f^* : A(Y) \to A(X)$ is a ring homomorphism. We've seen that this uniquely determines $f : X \to Y$ as follows: $f(x) = (f^*y_1, \ldots, f^*y_m)(x)$, so f is a morphism from X to Y. □

Definition 13. *An affine variety over k, an algebraically closed field, is a topological space X together with a sheaf $\mathcal{O}_X$ of k-valued functions on X such that $(X, \mathcal{O}_X)$ is isomorphic to an irreducible algebraic set together with its structural sheaf.*

Remark 5.2. *$(X, \mathcal{O}_X)$ is an affine variety ⇔ $\exists Y \subseteq \mathbb{A}^n$, an irreducible algebraic set, such that $(X, \mathcal{O}_X) \cong (Y, \mathcal{O}_Y)$, i.e. there exists a homeomorphism $f : X \to Y$, such that for all open sets $U \subseteq Y$, the pullback map $f_U^* : \mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}(U))$ is an isomorphism of rings and these isomorphisms f_U^* are compatible with restrictions.*

Note that if $X \subseteq \mathbb{A}^n$ is an irreducible algebraic set, then $(X, \mathcal{O}_X)$ is an affine variety, but there are many more examples of affine varieties.

Theorem 5.3. *Suppose $(X, \mathcal{O}_X)$ is an affine variety and $f \in A(X) = \mathcal{O}_X(X)$. Then the distinguished open set $(X_f, \mathcal{O}_{X_f})$ is an affine variety.*

X_f is open in X, since $X_f = X - Z(f)$, so for $U \subseteq X_f$ open, U is open in X and $\mathcal{O}_{X_f}(U) = \mathcal{O}_X(U)$.

Example 22. *Before we start to prove this theorem, we will explain the idea of the proof in a simple example. We consider the distinguished open set $\mathbb{A}^1 - \{0\} = \mathbb{A}^1_x$ and we realize this as an affine variety in $\mathbb{A}^2$. Let*
$Z(xy = 1) = V \subseteq \mathbb{A}^2$. Then projection $\pi : V \to \mathbb{A}^1_x$, $\pi(x, y) = x$, is an isomorphism from V to $\mathbb{A}^1_x$, thus $\mathbb{A}^1_x$ is an affine variety.

Proof. Suppose $X \subseteq \mathbb{A}^n$, and let $\mathfrak{a} = I(X)$ be the ideal of X. For $f \in A(X)$, let $\tilde{f} \in k[x_1, \ldots, x_n]$ be a representative for f, i.e. $\tilde{f} \mapsto f \bmod \mathfrak{a}$. We want to realize X_f as an affine variety in $\mathbb{A}^{n+1}$. We define the ideal

$$J = \mathfrak{a} + (1 + \tilde{f}(x_1, \ldots, x_n)x_{n+1}) \subseteq k[x_1, \ldots, x_{n+1}].$$

Claim: J is a prime ideal. Since

$$\frac{k[x_1, \ldots, x_{n+1}]}{J} = \frac{k[x_1, \ldots, x_n]}{\mathfrak{a}}\left[\frac{1}{f}\right] = A(X)_f$$

is a localization of the integral domain $A(X)$, the ring $A(X)_f$ must also be a domain. Thus J is prime.

Define

$$\begin{array}{ccc} & \pi & \\ Z(J) & \to & \mathbb{A}^n \\ (x_1, \ldots, x_{n+1}) & \mapsto & (x_1, \ldots, x_n). \end{array}$$

Then $\pi : Z(J) \to X_f$ is a homeomorphism. Note that

$$\begin{gathered}(x_1, \dots, x_{n+1}) \in Z(J) \Leftrightarrow \\ (x_1, \dots, x_n) \in Z(\mathfrak{a}) \subseteq \mathbb{A}^n \text{ and } \tilde{f}(x_1, \dots, x_n)x_{n+1} = 1 \Leftrightarrow \\ (x_1, \dots, x_n) \in X \text{ and } \tilde{f}(x_1, \dots, x_n) \neq 0 \Leftrightarrow \\ (x_1, \dots, x_n) \in X_f\end{gathered}$$

hence π establishes a bijection between $Z(J)$ and X_f. Proving π is a homeomorphism is left to the reader.

Claim: π establishes an isomorphism $(Z(J), \mathcal{O}_{Z(J)}) \cong (X_f, \mathcal{O}_{X_f})$. We need to show that $\forall U \subseteq X_f$ open, $\pi^* : \mathcal{O}_{X_f}(U) \to \mathcal{O}_{Z(J)}(\pi^{-1}(U))$ is an isomorphism. It is enough to check this for the distinguished open subsets of X_f, $U = X_g \cap X_f$, $g \in A(X)$. Fix $g \in A(X)$, then $U = X_g \cap X_f = X_{fg}$, and

$$\begin{aligned}\mathcal{O}_{X_f}(U) &= \mathcal{O}_X(X_{fg}) \\ &= A(X)_{fg} \\ &= \frac{k[x_1, \dots, x_n]}{\mathfrak{a}}\left[\frac{1}{fg}\right].\end{aligned}$$

Let $\tilde{g}$ be a representative for g, then $\pi^{-1}(U) = \pi^{-1}(X_g) = Z(J)_{\tilde{g}}$, and

$$\begin{aligned}\mathcal{O}_{Z(J)}(\pi^{-1}(U)) &= \mathcal{O}_{Z(J)}(Z(J)_{\tilde{g}}) \\ &= \mathcal{O}_{Z(J)}(Z(J))_{\tilde{g}} \\ &= \frac{k[x_1, \dots, x_{n+1}]}{J}\left[\frac{1}{\tilde{g}}\right] \\ &= \frac{k[x_1, \dots, x_n]}{\mathfrak{a}}\left[\frac{1}{f}\right]\left[\frac{1}{g}\right] \\ &= \frac{k[x_1, \dots, x_n]}{\mathfrak{a}}\left[\frac{1}{fg}\right]\end{aligned}$$

Thus $\pi : Z(J) \to X_f$ gives isomorphisms $\pi^* : \mathcal{O}_{X_f}(U) \to \mathcal{O}_{Z(J)}(\pi^{-1}(U))$, so $(X, \mathcal{O}_{X_f}) \cong (Z(J), \mathcal{O}_{Z(J)}) \Rightarrow (X_f, \mathcal{O}_{X_f})$ is an affine variety. □

Corollary 5.4. *Every affine variety X has a basis for the Zariski topology consisting of open affine subsets, $\{X_f | f \in A(X)\}$.*

Remark 5.5. *Although for X affine and $f \in A(X)$, the open set X_f is affine, it is not true that any open subset of an affine variety is affine.*

Example 23. *Let $X = \mathbb{A}^2$, $U = \mathbb{A}^2 - \{(0, 0)\}$. Then we have shown that U is not affine, although U is the union of the open affine sets $\mathbb{A}^2_x := \mathbb{A}^2 - \{x = 0\}$ and $\mathbb{A}^2_y := \mathbb{A}^2 - \{y = 0\}$.*

6 2/4 - Klonoff

Definition 14. *A ringed space is a pair $(X, \mathcal{O}_X)$, where X is a topological space and $\mathcal{O}_X$ is a sheaf of rings on X.*

Definition 15. *An isomorphism of ringed spaces* $(X, \mathcal{O}_X) \simeq (\Sigma, \mathcal{O}_\Sigma)$ *consists of a homeomorphism* $f : X \to \Sigma$ *together with ring isomorphisms*

$$f^* : \mathcal{O}_\Sigma(U) \to \mathcal{O}_X(f^{-1}(U))$$

for every open set $U \subset \Sigma$*, which commute with restrictions. (i.e induces an isomorphism of sheaves)*

Definition 16. *A ringed space* $(X, \mathcal{O}_X)$ *is an affine variety if there exists an irreducible algebraic set* $\Sigma \subset \mathbb{A}^n$ *and an isomorphism of ringed spaces* $(X, \mathcal{O}_X) \simeq (\Sigma, \mathcal{O}_\Sigma)$*, where* $\mathcal{O}_\Sigma$ *denotes the sheaf of regular functions on* Σ*.*

If X is an affine variety, it is not true that every open $U \subset X$ is an affine variety.

Example 24. *Let* $X = \mathbb{A}^2$*, and* $U = \mathbb{A}^2 - \{(0,0)\}$ *Then* U *is not an affine variety.*

Proof. Suppose that U is affine. Let $i : U \hookrightarrow \mathbb{A}^2$ be the inclusion map. Then i induces $i^* : \mathcal{O}_{\mathbb{A}^2}(\mathbb{A}^2) \to \mathcal{O}_U(U)$. We have the following identifications. $\mathcal{O}_{\mathbb{A}^2}(\mathbb{A}^2)$ is just the ring of polynomials in two variables $k[X, Y]$, and as we saw earlier $\mathcal{O}_U(U) = \mathcal{O}_{\mathbb{A}^2}(U) = k[X, Y]$. So i^* is just the identity map, an isomorphism. This says that $i : U \hookrightarrow \mathbb{A}^2$ is an isomorphism. This is a contradiction since i is not onto. □

Definition 17. *A connected topological space* X *with a sheaf of* k*-valued functions on* X*,* $\mathcal{O}_X$*, is a pre-variety if there is a finite open cover* $\{U_\alpha\}$ *of* X *such that the ringed space* $(U_\alpha, \mathcal{O}_X|_{U_\alpha} = \mathcal{O}_{U_\alpha})$ *is an affine variety.*

Remarks:

- $(X, \mathcal{O}_X)$ a pre-variety. If $U \subset X$ is open, then $(U, \mathcal{O}_X|_U)$ is also a pre-variety.
- If X is a pre-variety, then X is an irreducible space.

Proof. Homework □

Theorem 6.1. *If* X *is a pre-variety, then* X *is noetherian (descending chains of closed sets stabilize).*

Proof. Let

$$Z_1 \supset Z_2 \supset \ldots \supset Z_n \supset Z_{n+1} \supset \ldots$$

be a descending chain of closed sets.

There is a finite open cover of X, $\{U_\alpha\}$ such that $(U_\alpha, \mathcal{O}_{U_\alpha})$ is affine. It is sufficient to show that each sequence

$$U_\alpha \cap Z_1 \supset U_\alpha \cap Z_2 \supset , \ldots, \ \supset U_\alpha \cap Z_n \supset U_\alpha \cap Z_{n+1} \supset , \ldots,$$

stabilizes.

Let us fix now an element U_α of the affine covering of X. Since $U_\alpha \simeq \Sigma$ is affine, we may assume that $U_\alpha \subset \mathbb{A}^n$.

For each i we have that $U_\alpha \cap Z_i$ is closed in U_α hence it corresponds to an ideal $\mathfrak{a}_i \in A(\Sigma)$. Now this gives an increasing sequence of ideals in the noetherian ring $A(\Sigma)$. Thus this sequence stabilizes and so the corresponding sequence of closed sets stabilizes. □

Corollary 6.2. *Any prevariety X is quasi-compact (compact but not hausdorff).*

Proof. A set is quasi-compact if any open cover has a finite subcover. This is equivalent to the following: If $\{F_i\}$ is any collection of closed sets satisfying $\cap F_i = \emptyset$, then there is a finite collection of the F_i's satisfying $\cap_{k=1}^{N} F_{i_k} = \emptyset$.

To obtain a contradiction suppose that F_i is a collection of closed sets of X satisfying $\cap F_i = \emptyset$ but no finite collection $\{F_{i_k}\}$ satisfies $\cap_{k=1}^{N} F_{i_k} = \emptyset$. Then we can construct a decreasing sequence of closed sets that does not stabilize as follows:

Start with $F_1 \neq \emptyset$. Then there is an F_{i_2} such that $F_1 \cap F_{i_2} \neq \emptyset$ and $F_1 \supsetneq F_1 \cap F_{i_2}$. Similarly there is an F_{i_3} such that $F_1 \cap F_{i_2} \cap F_{i_3} \neq \emptyset$. So $F_1 \supsetneq F_1 \cap F_{i_2} \supsetneq F_1 \cap F_{i_2} \cap F_{i_3}$. Continuing in this way we construct a decreasing sequence of closed sets that does not stabilize. This is a contradiction, so there is a finite subcollection of F_i with empty intersection and X is thus quasi-compact. □

Enough abstract nonsense.

Example 25. *The projective line $\mathbb{P}^1$.*

The projective line, $\mathbb{P}^1$ is defined to be the quotient of $S = k^2 \setminus (0,0)$ by the equivalence relation $\sim$ where $x \sim y$ if $x = \lambda y$ for some $\lambda \in k^*$. So as a set, $\mathbb{P}^1 = S/\sim$.

We can write $\mathbb{P}^1 = \{[x,y] : (x,y) \neq (0,0)\}$.

In the familiar case of $k = \mathbb{C}$, we get the complex projective line $\mathbb{P}^1(\mathbb{C})$, which is homeomorphic to the sphere S^2, and can be thought of as the complex line union a point: $\mathbb{C} \cup \infty$.

Return to the general case where k is any (algebraically closed) field.

Let $U = \mathbb{A}^1$ with coordinate u, and set $U_0 = \mathbb{A}^1 \setminus \{0\}$. Similarly, set $V = \mathbb{A}^1$ with coordinate v, and define $V_0 = \mathbb{A}^1 \setminus \{0\}$. We want to glue U and V along U_0 and V_0 but we first have to specify the gluing recipe. We will glue via the isomorphism $f : U_0 \to V_0$ where $u \mapsto \frac{1}{u} = v$.

Claim that f is an isomorphism of pre-varieties. First note that the ring of functions on U_0 is $k[u]_u = k[u, \frac{1}{u}]$. Similarly the ring of functions on V_0 is the ring $k[v, \frac{1}{v}]$. The induced map $f^* : \mathcal{O}_{V_0}(V_0) \to \mathcal{O}_{U_0}(U_0)$ is given by $v \mapsto \frac{1}{u}, u \mapsto \frac{1}{v}$. It is clear that f is a k-algebra isomorphism.

We then use f to glue U and V along U_0 and V_0. As a set

$$\mathbb{P}^1 := U \amalg V/(u \in U_0 \sim \frac{1}{u} = v \in V_0)) = U \cup \{\infty\}.$$

Here $\infty \in \mathbb{P}^1$ denotes the equivalence class of $0 \in V$. We want to show that $\mathbb{P}^1$ is a pre-variety.

1. Give $\mathbb{P}^1$ a topology, the quotient topology, that is, a set $W \subset \mathbb{P}^1$ is open if and only if $W \cap U$ in open in U and $W \cap V$ is open in V.

2. We now endow $\mathbb{P}^1$ a sheaf of regular functions. (Next time)

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Last time we defined prevarieties. Now we will see a few examples:

Example 26. $\mathbb{P}^1$ *– the projective line.*
Let $U = \mathbb{A}^1$, $U_0 = \mathbb{A}^1 - \{0\}$, $V = \mathbb{A}^1$, $V_0 = \mathbb{A}^1 - \{0\}$. *We glue* U *and* V *together along* U_0 *and* V_0 *as follows: first we define an isomorphism*

$$\begin{gathered} f : U_0 \to V_0, f(u) = 1/u \\ f^* : \mathcal{O}_{V_0}(V_0) \xrightarrow{\sim} \mathcal{O}_{U_0}(U_0) \\ k[v, 1/v] \xrightarrow{\sim} k[u, 1/u] \\ v \mapsto 1/u, 1/v \mapsto u \end{gathered}$$

So $\mathbb{P}^1 = U \sqcup V/(u \sim 1/u)$ *is the identification. Set-theoretically we have that* $\mathbb{P}^1 = U \cup \{\infty\}$.

Example 27. *We now explain a more general procedure of gluing two prevarieties along open subsets: Assume that* U, V *are prevarieties,* $U_0 \subseteq U$ *and* $V_0 \subseteq V$ *are open subsets and*

$$f : (U_0, \mathcal{O}_{U_0}) \xrightarrow{\sim} (V_0, \mathcal{O}_{V_0}) \text{ is an isomorphism.}$$

Glue U *and* V *along* U_0 *and* V_0 *via* f.

$$X = U \sqcup V/U_0 \ni u \sim f(u) \in V_0$$

$$i : U \hookrightarrow X, i(u) = \overline{u} \text{ and } j : V \hookrightarrow X, j(v) = \overline{v} \text{ are inclusions.}$$

So $X = U \cup V$.

Define a topology on X :
$W \subseteq X$ *open* $\Leftrightarrow$
$i^{-1}(W) = W \cap U$ *open in* U, *and* $j^{-1}(W) = W \cap V$ *open in* V.

Functions on X : *"pairs of functions on* U *and* V *that agree on the overlap"*

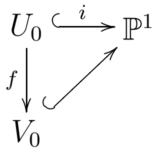
$$\begin{gathered} \mathcal{O}_X(W) = \{(\phi, \eta) \in \mathcal{O}_U(i^{-1}(W)) \times \mathcal{O}_V(j^{-1}(W)) : f^*(\eta) = \phi\} \\ f^* : \mathcal{O}_{V_0}(V_0 \cap j^{-1}(W)) \to \mathcal{O}_{U_0}(U_0 \cap j^{-1}(W)) \end{gathered}$$

$$\begin{array}{ccc} U_0 \cap i^{-1}(W) & \xhookrightarrow{\quad i \quad} & W \subset X \\ & {\scriptstyle f} \searrow \quad \nearrow {\scriptstyle j} & \\ & V_0 \cap j^{-1}(W) & \end{array}$$

commutes.

In this way, $(X, \mathcal{O}_X)$ *becomes a prevariety.*

Example 28. *Determine* $\Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1})$*, the global sections of* $\mathbb{P}^1$*.*



$$\begin{aligned} \phi \in \mathcal{O}_U(U), \eta \in \mathcal{O}_V(V) &\text{ such that } f^*(\eta) = \phi. \\ \mathcal{O}_U(U) = \Gamma(\mathbb{A}^1, \mathcal{O}_{\mathbb{A}^1}) = k[u], &\text{ and } \mathcal{O}_V(V) = k[v]. \\ \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) &= \{(\phi, \eta) \in k[u] \times k[v] : f^*(\eta) = \phi\} \\ &= \{(\phi, \eta) \in k[u] \times k[v] : \eta(1/u) = \phi(u)\} \end{aligned}$$

$$\text{But } \eta(1/u) = \phi(u) \Longrightarrow \phi = \eta \in k \text{ (degree 0)}$$
$$\Rightarrow \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = k$$

(So global functions on $\mathbb{P}^1$ *are constants.)*

Remark 7.1. $\mathbb{P}^1$ *is not affine.*
To see this we look at the inclusion $i : \{0\} \hookrightarrow \mathbb{P}^1$*. If* $\mathbb{P}^1$ *was affine, then since the map induced at the level of affine coordinate rings* $i^* : A(\mathbb{P}^1) \xrightarrow{\sim} A(0)$ *is an isomorphism, it would follow that* $i : \{0\} \hookrightarrow \mathbb{P}^1$ *is an isomorphism as well, which is clearly not the case.*

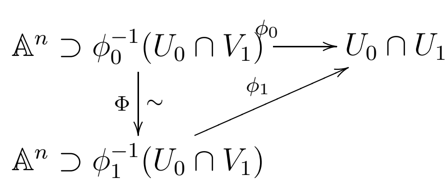
Example 29. *A pathological example.*

$$U = \mathbb{A}^1, U_0 = \mathbb{A}^1 - \{0\}, V = \mathbb{A}^1, V_0 = \mathbb{A}^1 - \{0\}$$
$$Y = U \sqcup V/\sim, \text{ where } u \in U_0 \sim u \in V_0$$
$$g : U_0 \to V_0, g(u) = u$$

This is "very" non-Hausdorff, because a sequence of points going to 0 has two limits at the double point 0.
$(Y, \mathcal{O}_Y)$ *is a prevariety, but it will not be a variety because of this bad double point.*

Example 30. *The* n*-dimensional projective space:*
As a set, $\mathbb{P}^n = k^{n+1} - \{(0, ..., 0)\}/\sim$*, where* $(a_0, ..., a_n) \sim (b_0, ..., b_n) \iff \exists \lambda \in k^*$ *such that* $b_i = \lambda a_i$*. We denote points in projective space by* $p = [a_0, ..., a_n] \in \mathbb{P}^n$ *(homogeneous coordinates of* p*. We want to turn* $\mathbb{P}^n$ *into a prevariety:*

Let $U_i = \{p = [a_0, ..., a_n] \in \mathbb{P}^n : a_i \neq 0\}$*, hence* $U_0 \cup \cdots \cup U_n = \mathbb{P}^n$*. Moreover, for each* $0 \leq i \leq n$ *we have a map* $\phi_i : \mathbb{A}^n \to U_i$ *where* $\phi_i(x_1, ..., x_n) = [x_1, ..., x_{i-1}, 1, x_i, ..., x_n]$*. This is a bijection, which we will use to glue* $U_0, U_1, ..., U_n$*.*



$\phi_0(a_1, ..., a_n) = [1, a_1, a_2, ..., a_n]$
$\phi_1(b_1, ..., b_n) = [b_1, 1, b_2, ..., b_n]$
So $\phi_0^{-1}(U_0 \cap U_1) = \{a_1 \neq 0\}$ *and* $\phi_1^{-1}(U_0 \cap U_1) = \{b_1 \neq 0\}$

We will define Φ *and show it is an algebraic map.*
We want $(a_1, ..., a_n) \mapsto (b_1, ..., b_n)$ *such that* $[1, a_1, a_2, ..., a_n] = [b_1, 1, b_2, ..., b_n]$
i.e. such that $\exists\ \lambda \in k^*$ *such that* $b_1 = \lambda,\ 1 = \lambda a_1,\ b_2 = \lambda a_2,\ \cdots b_n = \lambda a_n$
$\Rightarrow \lambda = 1/a_1$
$\Rightarrow b_1 = 1/a_1, b_2 = a_2/a_1, \cdots, b_n = a_n/a_1.$
So $\Phi(a_1, ..., a_n) = (1/a_1, a_2/a_1, ..., a_n/a_1)$

Glue U_0 *and* U_1 *via* Φ. *Then we need to show that we can glue* U_0, U_1 *with* U_2 *in any order and get the same, etc. This shows that* $\mathbb{P}^n$ *is a prevariety.*

Example 31. $\mathbb{P}^4$

$$[1 : 2 : 0 : 3] \in U_0 \cap U_1 \cap (U_3 - U_2)$$

We can write

$$\begin{array}{ccccc} [1 : 2 : 0 : 3] & = & [1/2 : 1 : 0 : 3/2] & = & [1/3 : 2/3 : 0 : 1] \\ \in Im(\phi_0) & & \in Im(\phi_1) & & \in Im(\phi_3) \end{array}$$

Exercise: Show $\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}) = k$
(essentially the same proof as for $\mathbb{P}^1$*, with more variables.)*

Function field of a prevariety :
Let X *be a prevariety. We define an equivalence relation on the set of pairs* (U, f)*, where* $\emptyset \neq U \subseteq X$ *open, and* $f \in \mathcal{O}_X(U)$ *by:*

$$(U, f) \sim (V, g) \iff \exists\ \emptyset \neq W \subseteq V \cap U, W \text{ open, such that}$$

$$res_{v,w}(f) = res_{v,w}(g) \text{ i.e. } f|w = g|w$$

(That is, two functions coincide on small open set.)

Note: Since X *is irreducible, any two non-empty open subsets* $U, V \subseteq X$ *intersect, so this definition makes sense. (We can also check that this is indeed an equivalence relation.)*

$k(X) := \{< U, f >\}$ *– the set of equivalence classes.*

Note:
1. This is very similar to the definition of a stalk
2. Think of analogy with meromorphic functions on $\mathbb{C}$*.*

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Proposition. *Let X and Y be prevarieties with $f : X \to Y$ any map. If $\exists \{V_i\}$ a finite affine covering of Y and $\{U_i\}$ an affine open covering of X such that $f(U_i) \subseteq V_i$ and $f^* : \mathcal{O}_Y(V_i) \to \mathcal{O}_X(U_i)$ for all i, then f is a morphism.*

Proof: Given last time.

Thus, given two prevarieties X, Y, and a continuous map $f : X \to Y$ with $(X, \mathcal{O}_X)$, to chech that f is an algebraic morphism it suffices to find open affine coverings $\{U_i\} \subseteq X$ and $\{V_i\} \subseteq Y$, with $f(U_i) \subseteq V_i$ and such that $f^* : \mathcal{O}_Y(V_i) \to \mathcal{O}_X(U_i)$ for all i, that is, f pulls back regular functions on V_i to regular functions on U_i.

Example: *Take $\pi : \mathbb{P}^2 - \{[0,0,1]\} \to \mathbb{P}^1$. Then*

$$\pi[x, y, z] = [x, y]$$

and since $\mathbb{P}^2 = \mathbb{A}^2 \cup (\mathbb{P}^1)_\infty$ then $\mathbb{A}^2 \hookrightarrow \mathbb{P}^2$ with $(x, y) \overset{\infty}{\mapsto} [x, y, 1]$. Since $\mathbb{P}^1 \hookrightarrow \mathbb{P}^2$ and we define $\mathbb{P}^1_\infty = \{[x, y, 0] : [x, y] \in \mathbb{P}^1\}$, then we are mapping from the origin to the line at infinity.

To check that π is a morphism, find an affine covering of $\mathbb{P}^1 = V_0 \cup V_1$ with coordinates $\{[u, v]\}$ and

$$V_0 = \{u \neq 0\} \subseteq \mathbb{P}^1$$
$$V_1 = \{v \neq 0\} \subseteq \mathbb{P}^1.$$

As $V_0 \simeq \mathbb{A}^1$ implies $[1, t] \mapsto t$ and $V_1 \simeq \mathbb{A}^1$ implies $t \mapsto [t, 1]$, we take

$$\pi^{-1}(V_0) = U_x = \{[x, y, z] : x \neq 0\} \simeq \mathbb{A}^2$$

and

$$\pi^{-1}(V_1) = U_y = \{[x, y, z] : y \neq 0\} \simeq \mathbb{A}^2,$$

so for $U_x \simeq \mathbb{A}^2$ and $V_0 \simeq \mathbb{A}^2$ the map in charts implies that $\mathbb{A}^2 \to \mathbb{A}^1$ and $U_x \xrightarrow{\pi} V_0$. This leads to an algebraic morphism of affine varieties as $[1, a, b] \overset{\pi}{\mapsto} [1, a]$, $[1, a] \mapsto [a]$, $[a, b] \mapsto [a]$ and $(a, b) \mapsto [a]$.

Now you can check it for the isomorphism $U_y \to V_1$ by again getting $(a, b) \mapsto a$ and recalling that $U_x \cup U_y = \mathbb{P}^2 - \{[0, 0, 1]\}$.

Example: *Recall as motivation that complex projective varieties are compact in the Euclidean topology. Consider the projective closure of the affine curve $X_0 : \{y = x^2\} \subseteq \mathbb{A}^2$ again with $\mathbb{A}^2 \cup (\mathbb{P}^1)_\infty \subseteq \mathbb{P}^2$ and an inclusion $i : \mathbb{A}^2 \hookrightarrow \mathbb{P}^2$ so that $i(x, y) = [x, y, 1]$. Take the projective closure of the parabola X_0 in $\mathbb{P}^2$ as $\overline{i(X_0)} = X$. We have that $x \mapsto [x, x^2, 1]$ and for the change of coordinates $[u, v, w]$ this implies that $u^2 = vw$ and so $X = \{[u, v, w] \in \mathbb{P}^2 : u^2 = vw\}$. The parabola intersects the line at infinity in a point with multiplicity two. Generally for $S = k[x_0 \ldots x_n]$ with $S_d = \{f$ the set of homogeneous polynomials of $\deg d\}$ we have $S = \bigoplus_{d \geq 0} S_d$.*

We would like to extend the correspondence between affine algebraic sets and ideals in polynomial rings to the case of projective space:

Definition: *A homogeneous ideal in $S = k[x_0, \ldots, x_n]$ is an ideal $\mathfrak{a}$ generated by homogeneous polynomials.*

Remark: *$\mathfrak{a} \subseteq S$ homogeneous $\Leftrightarrow \mathfrak{a} = \bigoplus_{d\geq 0}(\mathfrak{a} \cap S_d)$. We associate to any homogeneous ideal $\mathfrak{a} \subseteq S$, the set $Z(\mathfrak{a}) = \{p \in \mathbb{P}^n : f(p) = 0 \quad \forall \quad f$ homogeneous in $\mathfrak{a}\} \subseteq \mathbb{P}^n$.*

Theorem. *The Zariski closed sets in $\mathbb{P}^n$ (defined a prevariety by the gluing of $n+1$ open affine pieces) are precisely the sets of the form $Z(\mathfrak{a})$ with $\mathfrak{a} \subset S$ being a homogeneous ideal.*

Proof: *Recall $\mathbb{P}^n = U_0 \cup \ldots \cup U_m$ for open affine $U_i \simeq \mathbb{A}^m$ where $U_i = \{p = [x_0 \ldots x_n] : x_i \neq 0\}$. $X = Z(\mathfrak{a})$ closed and $\mathfrak{a} \subseteq S$ homogeneous. Then $X = Z(\mathfrak{a})$ closed $\Leftrightarrow \forall i \quad X \cap U_i$ is closed in U_i. Check for $i = 0$, g a homogeneous polynomial. $X \cap U_0 = \{p = [1, x_1 \ldots x_n] : \forall g \in \mathfrak{a}, g(1, x_1 \ldots x_n) = 0\}$. Let $\tilde{g} = \{g(1, x, \ldots x_n = 0)\}$ with $\tilde{g} \in k[y, \ldots, y_n$. Then $X \cup U_0 = Z(\langle \tilde{g} : g \in \mathfrak{a}\rangle)$ is closed in U_0. Conversely, say $X \subseteq \mathbb{P}^n$ is closed, and assume that X is irreducible by the Noetherian condition. Let $X \neq 0$ and $X \cap U_0 \neq \emptyset$. Since $U_0 \cap X$ is closed in U_0 we have that $U_0 \cap X = Z(\tilde{\mathfrak{a}})$ where $\tilde{\mathfrak{a}}$ is an ideal in $k[y_1 \ldots y_n]$. Now, homogenize every polynomial in $\tilde{\mathfrak{a}}$ so that $g(y_1 \ldots y_n) \to G(x_0 \cdots x_n) = x_0^d g\left(\frac{x_1}{x_0} \cdots \frac{x_n}{x_0}\right)$. For example, with the given polynomial*

$$y_2^2 + 2y_2 \to \left[\left(\frac{x_1}{x_0}\right)^2 + \frac{2x_2}{x_0}\right] x_0^2 = x_1^2 + 2x_2x_0.$$

Claim: *$Z(\tilde{\mathfrak{a}}) = X$ with $\tilde{\mathfrak{a}}$ the ideal in $k[x_0 \cdots x_n]$ generated by all G's.*

It is clear that $X \cap V_0 \in Z(\mathfrak{a})$ by all the G's, so we must have that $X \subseteq Z(\mathfrak{a})$. To prove the reverse take $X = \bigcup_{i=0}^n (X \cap U_i)$ which implies that $X = Z(\mathfrak{a})$.

Projective Nullstellensatz: *We have associated to each homogeneous ideal $\mathfrak{a} \subset k[x_0, \ldots, x_n]$ the algebraic set $Z(\mathfrak{a}) \subseteq \mathbb{P}^n$. Conversely, to each set $X \subseteq \mathbb{P}^n$, we associate the ideal $I(X)$ the ideal generated by all homogeneous polynomials $f \in k[x_0 \cdots x_n]$ such that $f|_X = 0$. Then there is a $1:1$ correspondence given by $\{$ closed algebraic sets in $\mathbb{P}^n\} \overset{Z}{\leftrightarrows} \{\mathfrak{a}$ radical homogeneous ideals in $k[x_0 \cdots x_n]$ which are different from $(x_0 \cdots x_n)\}$.*

Note: $I(\emptyset) = S$. Note also that $(x_0 \ldots x_n)$ is not the ideal of any algebraic set $X \subseteq \mathbb{P}^n$.

Homogeneous Coordinate Ring: *For an algebraic set $X \subseteq \mathbb{P}^n$, if $I(X)$ is the corresponding homogeneous ideal, then the homogeneous coordinate ring is defined by $S(X) := S/I(X)$.*

Remark: *$S(X)$ unlike the affine coordinate ring of an affine variety, does not determine the variety. For instance taking $X \subseteq \mathbb{P}^n$ and $Y \subseteq \mathbb{P}^m$, an isomorphism between between X and Y does not imply that $S(X)$ and $S(Y)$ are isomorphic.*

Example: *Take $X = \mathbb{P}^1$ and the conic $Y = \{x^2 + y^2 = z^2\} \subseteq \mathbb{P}^2$. For some point $p \in Y$ consider the projection $\pi_p : Y \to X$ as an isomorphism. We have that $S(X) = k[u, v]$ and $S(Y) = k[x, y, z]/(x^2 + y^2 - z^2)$. If we look at the k-vector space given by $S(X)_1 = ku \oplus kv$ and $S(Y)_1 = kx \oplus ky \oplus kz$ we notice the dimensions are different, so $S(X)$ is not isomoprphis to $S(Y)$ although X is isomorphic to Y.*

9 Projective Prevarieties Cont. - 2/16 - Patthak

9.1 Projective Subvarieties

Previously we have seen how to construct projective spaces $\mathbb{P}^n$ from affine spaces, i.e., considering $n+1$-affine pieces $U_0, \cdots, U_n$ and defining an appropriate identification map. Alternatively, we can directly define the Zariski topology on projective spaces algebraically, i.e., by taking the closed sets to be the zero locus of a homogeneous ideal (i.e., a graded ideal) generated by homogeneous polynomials. The later method has the additional advantage that it no longer requires the identification map explicitly. This is an convenient approach and this is the approach we will try to follow.
We next define Projective subvarieties.

Definition 18. *Let $(X, \mathcal{O}_X)$ be a prevariety and $Y \subseteq X$ be a closed set. We define an induced prevariety on Y as follows:*

Claim 9.1. *$(Y, \mathcal{O}_Y)$ is a prevariety.*

Proof. Note that if $X = \cup_\alpha V_\alpha$ be a finite open covering, where V_α are affine, then it follows that $Y = \cup_\alpha (Y \cap V_\alpha)$. Therefore, it is enough to show that $(Y_\alpha = Y \cap V_\alpha, \mathcal{O}_{Y_\alpha})$ is an affine variety. We need to check that the sheaves are equal which is straightforward (See Mumford, Proposition 3, §4, Chapter I, page 24). □

Remark 9.2. *Note that from the above it also follows that if $U \subseteq X$ is open, then $\forall U \subseteq Y$, $\mathcal{O}_Y(U) = \mathcal{O}_X(U)$.*
Also it is possible to show that if $(X, \mathcal{O}_X)$ is a prevariety and $Y \subseteq X$ is locally closed[2], then $(Y, \mathcal{O}_Y)$ is a prevariety.
The set of all prevarieties obtained this way are called the sub-prevarieties *of X.*

Example 32. *If $X \subseteq \mathbb{P}^n$ closed, then $(X, \mathcal{O}_X)$ (i.e., induced from $\mathbb{P}^n$) is a prevariety.*

Recall that for all affine $X \subseteq \mathbb{A}^n$, we defined previously that Alternatively, we can also define $\mathcal{O}_X(U)$ induced from $\mathbb{A}^n$.

9.2 Projective Closure

Let $X \subset \mathbb{A}^n$ be an affine prevariety. Also let $I(X) = \mathcal{J} \subseteq k[y_1, \cdots, y_n]$. Then $(y_1, \cdots, y_n) \mapsto [1, y_1, \cdots, y_n]$ defines a natural inclusion map $\imath : U_0 \simeq \mathbb{A}^n \hookrightarrow \mathbb{P}^n$.

Definition 19. *Let $X \subseteq \mathbb{A}^n$ be affine. Then we define the projective closure of X, denoted $\overline{X}$, to be $\overline{X} = \overline{I(X)} \subseteq \mathbb{P}^n$ in the Zariski topology.*

Given any $f \in k[y_1, \cdots, y_n]$, there is a canonical homogenization of f in the graded homogeneous ring defined by degree of f.

Claim 9.3. *Let $\mathfrak{a}$ be the ideal generated by homogenizing all $f \in \mathcal{J} = I(X)$, i.e., It holds then that $\overline{X} = Z(\mathfrak{a})$.*

[2]Definition: A subset W of a topological space X is said to be is locally closed if every point p in W has an open neighbourhood U in X such that $W \cap U$ is closed in U.

Proof. Following the definition of canonical homogenization, it is clear that $X \subseteq Z(\mathfrak{a})$. Then the defintion of projective closure implies that $\overline{X} \subseteq Z(\mathfrak{a})$.
Therefore it remains to show that $\overline{X}^c \subseteq Z(\mathfrak{a})^c$ where A^c defines the compliment of A. Let $p \in \overline{X}^c$. This implies that $\exists F \in S$, such that equivalently, $p \notin Z(\mathfrak{a})$. This completes the proof. □

We stress that the above claim need not be true when restricted to the generating set of polynomials of the ideal $\mathcal{J}$. To illustrate we consider the following example.

Example 33. *Twisted cubic: Let* $X = \{(t, t^2, t^3) \ : \ t \in k\} \subseteq \mathbb{A}^3$. $f_1 = y_1^2 - y_2$ *and* $f_2 = y_1^3 - y_3$ *and denote their homogenization by* $F_1 = y_1^2 - y_0 y_2$ *and* $F_2 = y_1^3 - y_3 y_0^2$. *We claim*[3] *that consider the closure of* $\overline{X} \subseteq \mathbb{P}^n$. *Clearly Consider the homogenized polynomial* $F = y_2^2 - y_1 y_3$. *Clearly Consider* $p = [0, 0, 1, 0]$. *Note that* $F(p) \neq 0$. *However,* $F_1(p) = F_2(p) = 0$.

9.3 Cones over an Affine Curve

Definition 20. *Let* $X \subseteq \mathbb{P}^n$ *be a nonempty projective set. The cone over* X*, denoted* $C(X)$*, is defined as follows: we first identify a point* $O = (0, \cdots, 0)$ *in* $\mathbb{A}^{n+1}$ *as origin. We then define a map* $\phi : \mathbb{A}^{n+1} - \{O\} \to \mathbb{P}^n$ *by* $(y_0, \cdots, y_n) \mapsto [y_0, \cdots, y_n]$ *i.e., that sends the affine coordinates to the homogeneous coordinates. Then the affine cone over* X *is defined to be The affine ideal of* $C(X)$ *is just* $\mathfrak{a} \subseteq k[y_0, \cdots, y_n]$ *homogeneous ideal of* X*, but* no longer graded *and no longer viewed as a homogeneous ideal.*

9.4 Projective Prevarieties as Ringed Spaces

Let $X \subseteq \mathbb{P}^n$ be a projective prevariety and $\mathfrak{a} \subseteq k[y_0, \cdots, y_n]$ be the correspoding homogeneous ideal. We have seen that X can be written as an union of $(n+1)$ affine pieces, seen how to get a projective prevariety $(X, \mathcal{O}_X)$ from the affine prevarieties $(X_i, \mathcal{O}_{X_i})$ and the identification maps. Let
Our goal is now to define the projective variety without having a need to define the identification maps. To that direction we far carries over nicely. Recall that $S = \oplus_{d \geq 0} S_d$ be the graded ring in $k[y_0, \cdots, y_n]$. Denote $S(X) = S/I(X)$ and $S_d(X) = S(X) \cap S_d$.

Definition 21. *We define the projective function field of* X *functions at* $p \in X$ *For* $U \subseteq X$ *open, we also define*

Remark 9.4.

Theorem 9.5. *prevariety. Furthermore,*

indeed a prevariety. Therefore we only prove the second part. The proof is divided into two parts. First we prove that Let $X_0 \subseteq \mathbb{A}^{n+1}$ be the affine piece corresponding to $y_0 \neq 0$. Recall that We will use the correspondence We note here that $g \neq 0 \implies G \neq 0$. Next we show that Note that this implies that since $g(p) \neq 0 \implies G(p) \neq 0$. □

[3] By an ideal generated by homogenized polynomials we mean the graded ideal in $\oplus_{d \geq 0} S_d$.

10 2/23 - Conceicao

Proposition 10.1. *Let $X \subset \mathbb{P}^n$ be a projective variety. Let $f_0, ..., f_s \in k[X_0, ..., X_n]$ homogeneous of the same degree such that $f_0, ..., f_s$ do not vanish simultaneously on X. Then $f : X \longrightarrow \mathbb{P}^s$ defined by $f(x) = [f_0(x) : ... : f_s(x)]$ is an algebraic morphism.*

Example 34. *The following example shows the non-validity of the reciprocal: not every morphism $X \longrightarrow \mathbb{P}^n$ can be given uniformly by polynomials. Let $f : \mathbb{P}^1 \longrightarrow \mathbb{P}^2$ be defined by $f[u : v] = [u^2 : uv : v^2]$. Set $X = \operatorname{Im} f$. Then $I(X) = (X_1^2 - X_0X_2)$ and if $P = [x_0 : x_1 : x_2] \in X$ then $x_1 \neq 0$ or $x_2 \neq 0$ or $P = [1 : 0 : 0]$*

11 Products of prevarieties

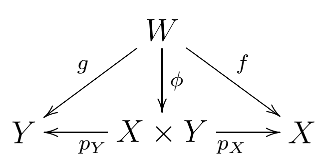
Let X and Y be prevarieties. We want the product $X \times Y$ to be a prevariety too. Unfortunately, we cannot just put the product topology in it as the following examplse shows.

Example 35. $\mathbb{A}^1 \times \mathbb{A}^1$*, as a prevariety should be isomorphic to $\mathbb{A}^2$. It's not hard to see that the Zariski topology in $\mathbb{A}^2$ is different from the product topology in $\mathbb{A}^1 \times \mathbb{A}^1$*

The key fact in defining the product of two varieties is that the product can be defined categorically. First let's see how to define the product in the category of sets.

Let's consider two sets X and Y. Let $p_X : X \times Y \longrightarrow X$ and $p_Y : X \times Y \longrightarrow Y$ be the canonical projections. Then $X \times Y$ satisfies:

For any set W with maps $W \xrightarrow{f} X$ and $W \xrightarrow{g} Y$ there's a unique map $\phi : W \longrightarrow X \times Y$ that makes



into a commutative diagram, i.e., you can factor f and g through the map ϕ, $f = p_X \circ \phi$ and $g = p_Y \circ \phi$

Remark 11.1. *1. Check that in the category of topological spaces, the product topology satisfies this universal property.*

2. If a product exists, it's unique up to isomorphism.

Now let's see how this universal property will give us a clue on how to define the product of two affine prevarieties.

Example 36. *Consider prevarieties $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$. Let's suppose that $X \times Y \subseteq \mathbb{A}^{m_n}$ is defined. Suppose $I(X) = (f_1, ..., f_s) \subset k[X_1, ..., X_n]$ and $I(Y) = (g_1, ..., g_t) \subset k[Y_1, ..., Y_m]$. Working out a few examples it's not hard to guess that $I(X \times Y) = (f_i, g_j) \subset k[X_1, ..., X_n, Y_1, ..., Y_m]$.*

The canonical projections $p_X : X \times Y \longrightarrow X$ and $p_Y : X \times Y \longrightarrow Y$ are morphisms that induce k-algebra homomorphisms

$$\begin{array}{ccc} A(X) & \longrightarrow & A(X \times Y) \\ & & \uparrow \\ & & A(Y) \end{array}$$

The commutative diagram in the category of affine varieties induces the following commutative diagram over the category of k-algebras:

$$\begin{array}{ccccc} A(X) & \xrightarrow{p_X^*} & A(X \times Y) & \xleftarrow{p_X^*} & A(Y) \\ & \searrow^{f^*} & \downarrow{\phi^*} & \swarrow^{g^*} & \\ & & W & & \end{array}$$

Where $\phi : W \longrightarrow X \times Y$ is the unique morphism obtained from the universal property of products. This last diagram is the diagram that defines Tensor products as Universal objects, so $A(X \times Y) = A(X) \otimes_k A(Y)$. Note that $A(X) \otimes_k A(Y)$ is a k-algebra and an integral domain, since $A(Y)$ and $A(X)$ are also integral domains. So the affine variety with ring of functions $A(X) \otimes_k A(Y)$ is the product $X \times Y$ of the affine varieties X and Y. Notice that the canonical maps $A(X) \longrightarrow A(X) \otimes_k A(Y)$ and $A(Y) \longrightarrow A(X) \otimes_k A(Y)$ corresponds to the canonical projections on X and Y respectively. Also it's not hard to see that

$$\begin{aligned} A(X \times Y) &= A(X) \otimes_k A(Y) = \frac{k[X_1, ..., X_n]}{(f_1, ..., f_s)} \otimes_k \frac{k[Y_1, ..., Y_m]}{(g_1, ..., g_r)} \\ &\simeq \frac{k[X_1, ..., X_n, Y_1, ..., Y_m]}{(f_i, g_j)} \end{aligned}$$

Example 37. *Let $X \subseteq \mathbb{A}^3$ be defined by $XYZ = X^2 + 3$ and $Y \subset \mathbb{A}^2 : st = 7$ then $X \times Y$ is the subset $\{(x, y, z, s, t) \in \mathbb{A}^5 | xyz = x^2 + 3, st = 7\}$*

12 Topology on $X \times Y$

Suppose X and Y affine varieties. The product of them as a set is just the set $X \times Y$. Let's see how to define the topology to get an affine variety. Let's start by defining the distinguished sets. For that pick $f \in A(X) \otimes_k A(Y)$. Recall that $f = \sum_i f_i(\underline{x}) \otimes g_i(\underline{y})$ and using the isomorphism

$$\frac{k[\underline{X}]}{I(X)} \otimes_k \frac{k[\underline{Y}]}{I(Y)} \longrightarrow \frac{k[\underline{X}, \underline{Y}]}{(I(X), I(Y))} \qquad f \otimes g \longmapsto f(\underline{x})g(\underline{y})$$

We see that

$$(X \times Y)_f = \{(x, y) \in X \times Y | \sum f_i(x)g_i(y) \neq 0\}$$

forms a base for the topology on $X \times Y$.

13 Field of functions of $X \times Y$

We know that $k(X \times Y)$ is by definition quot(A(X) $\otimes_k$ A(Y)) which is equal to $k(X) \otimes_k k(Y)$.

14 Stalks

In this case, the stalk at a point $(x, y) \in X \times Y$ is given by

$$\mathcal{O}_{X\times Y,(x,y)} = (A(X) \otimes_k A(Y))_{m_{(x,y)}}$$

Recall that in general $\mathcal{O}_{X\times Y,(x,y)}$ = ring of germs of functions $X \times Y$ localized at the ideal of those functions vanishing at (x, y). Let's prove that $\mathcal{O}_{X\times Y,(x,y)} \simeq (\mathcal{O}_{X,x} \otimes_k \mathcal{O}_{Y,y})_{m_x \otimes \mathcal{O}_{Y,y} + \mathcal{O}_{X,x} \otimes m_y}$.

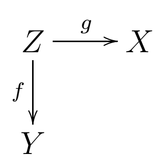
Indeed, let $J = m_x \otimes \mathcal{O}_{Y,y} + \mathcal{O}_{X,x} \otimes m_y$, i.e.,

$$J = \left\{ \sum_i (f_i \otimes g_i + u_i \otimes v_i) \middle| f_i(x) = g_i(y) = 0, g_i, v_i \in \mathcal{O}_{Y,y}, f_i, u_i \in \mathcal{O}_{X,x} \right\}$$

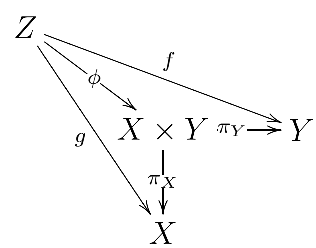
Notice that for $s = \sum_i f_i \otimes g_i \in \mathcal{O}_{X,x} \otimes_k \mathcal{O}_{Y,y}$ with $\sum_i f_i(x) g_i(y) = 0$, we'll have $s = \sum (f_i - a_i) \otimes g_i + \sum a_i \otimes (g_i - b_i) \in (m_x \otimes \mathcal{O}_{Y,y} + \mathcal{O}_{X,x} \otimes m_y)$, where $b_i = g_i(y)$ and $a_i = f_i(x)$ are elements of k. Therefore J is the ideal of germs on $X \times Y$ vanishing at (x, y).

Now we are ready to discuss the general case:

Theorem 14.1. *If X and Y are prevarieties defined over k then their product $X \times Y$ exists (as a prevariety satisfying the universal property: Given*



There exists a unique morphism $\phi = (f, g) : Z \longrightarrow X \times Y$ making the following diagram comutative



Proof: As a set $X \times Y = \{(x, y) : x \in X, y \in Y\}$. We define the *topology* in the following way: Let $\mathcal{U} \subseteq X$ and $\mathcal{V} \subseteq Y$ be affine open subsets. Let $f = \sum_i f_i \otimes g_i$, $f_i \in A(\mathcal{U}) = \mathcal{O}_X(\mathcal{U})$, $g_i \in A(\mathcal{V}) = \mathcal{O}_Y(\mathcal{V})$. A basis for the topology in $X \times Y$ is given by the sets

$$\left\{ (x, y) \in \mathcal{U} \times \mathcal{V} \middle| \sum_i f_i(x) g_i(y) \neq 0 \right\}$$

over all open affines $\mathcal{U} \subseteq X$, $\mathcal{V} \subseteq Y$, $f_i \in \mathcal{O}_X(\mathcal{U})$, $g_i \in \mathcal{O}_Y(\mathcal{V})$.

Define:

the function field by

$$k(X \times Y) := k(X) \otimes_k k(Y)$$

the stalks by

$$\mathcal{O}_{X\times Y,(x,y)} := (\mathcal{O}_{X,x} \otimes_k \mathcal{O}_{Y,y})_{m_x \otimes \mathcal{O}_{Y,y} + \mathcal{O}_{X,x} \otimes m_y}$$

and *the regular functions* on an open set $\mathcal{U} \subseteq X \times Y$ by

$$\mathcal{O}_{X\times Y}(\mathcal{U}) := \bigcap_{(x,y)\in\mathcal{U}} \mathcal{O}_{X\times Y,(x,y)}$$

Using this definitions it's not hard to show that $(X \times Y, \mathcal{O}_{X\times Y})$ is a prevariety. Now we're left to check that $X \times Y$ satisfies the universal property of a product. To check that let us consider for any prevariety Z and morphisms $s : Z \longrightarrow Y$, $r : Z \longrightarrow X$ the following diagram:

$$\begin{array}{ccc} Z & \xrightarrow{\ r\ } & X \\ {\scriptstyle s}\downarrow & & \uparrow{\scriptstyle p_X} \\ Y & \xleftarrow[p_Y]{} & X \times Y \end{array}$$

We want to construct a morphism $\phi : Z \longrightarrow X \times Y$ which makes the diagram commutes.

Let $\phi = (r, s) : Z \longrightarrow X \times Y$ be the map defined by $z \longmapsto \phi(z) = (r(z), s(z))$. If $\mathcal{U} \subseteq X$ and $\mathcal{V} \subseteq Y$ are non-empty affine open sets, then define $Z_{\mathcal{U},\mathcal{V}} = r^{-1}(\mathcal{U}) \cap s^{-1}(\mathcal{V})$ which is an open set in Z. It suffices to check that $\phi|_{Z_{\mathcal{U},\mathcal{V}}}$ is a morphism. To show that $Z_{\mathcal{U},\mathcal{V}} \xrightarrow{\phi} \mathcal{U} \times \mathcal{V}$ is a morphism is equivalent to show that $\phi^* : A(\mathcal{U} \times \mathcal{V}) \longrightarrow \mathcal{O}_Z(Z_{\mathcal{U},\mathcal{V}})$ is a homomorphism. Recall that we can identify $A(\mathcal{U}\times\mathcal{V})$ with $A(\mathcal{U}) \otimes_k A(\mathcal{V})$, since $\mathcal{U}$ and $\mathcal{V}$ are affine prevarieties. So take $f(x) \otimes g(y) \in A(\mathcal{U} \times \mathcal{V})$. Then it's not hard to see that $\phi^*(f \otimes g) = f^*(f)s^*(g)$ and that this is a homomorphism. Also, since r and s are morphism, we have that $r^*(f)$ and $s^*(g)$ are regular functions, as well as their product. So ϕ is a morphism and clearly satisfy the universal property.

Example 38. *Segre Embedding*

Let's make $\mathbb{P}^n \times \mathbb{P}^m$ into a projective prevariety. Consider $\phi : \mathbb{P}^n \times \mathbb{P}^m \longrightarrow \mathbb{P}^{mn+m+n}$ the map defined by

$$\phi([x_0 : \ldots : x_n], [y_0 : \ldots : y_m]) = [z_{ij} = x_i y_j]_{\substack{0\le i\le n \\ 0\le j\le m}}$$

We should notice that this map can be represented as the matrix:

$$A = \begin{bmatrix} z_{00} & \cdots & z_{0m} \\ \vdots & \ddots & \vdots \\ z_{n0} & \cdots & z_{nm} \end{bmatrix}$$

ϕ is well-defined since it clearly does not depend on the homogeneous coordinate and you always can find i and j such that $x_i y_j \neq 0$. Also ϕ is injective as one can easily verify.

Let $Z = \mathrm{Im}(\phi)$. Then Z is a closed algebraic set in $\mathbb{P}^{mn+m+n}$. In fact, $Z = \{A = [z_{ij}]_{\substack{0\le i\le n \\ 0\le j\le m}} \mid \mathrm{rank}(A) = 1\}$, that is, $I(Z)$ is generated by the 2×2 minors $z_{ij}z_{lk} - z_{lj}z_{ik}$. One can check that ϕ gives an embedding $\phi : \mathbb{P}^n \times \mathbb{P}^m \xhookrightarrow{\phi} \mathbb{P}^{mn+m+n}$ (see Munford's, The Red Book of Varieties and Schemes, pg 37).

15 2/25

15.1 Varieties

Proposition 15.1. *Let X be a variety with open affine subsets U and V. Then $U \cap V$ is affine and*

$$O_X(U \cap V) = O_X(U) \cdot O_X(V) \subset k(X).$$

This is the compositum, the smallest subring of $k(X)$ containing both $O_X(U)$ and $O_X(V)$.

Remark 15.2. *This is not another criterion for determining whether something is a variety. For example, in the line with doubled origin, the intersection of the two glued affine pieces is $\mathbb{A}^1 \setminus \{0\}$, which is affine, being isomorphic to $xy - 1 = 0$ in $\mathbb{A}^2$. However, the result can be used to show that prevarieties are not varieties. Forexample, if we generalize our cunstruction to $\mathbb{A}^2$ with a doubled origin, then the intersection is $\mathbb{A}^2 \setminus \{0\}$, which is not affine. Therefore this prevariety is not a variety.*

15.2 Rational Maps

Let X be a variety, Y a prevariety, and U an open subset of Y. If $f_U : U \to X$ is an algebraic morphism, can we extend it to a morphism $f : Y \to X$?

Last time, we saw that the projective closure of $Y : y = x^2$ was isomorphic to $\mathbb{P}^1$ via the projection map from a point. Stated carefully, $\pi_p : Y \setminus \{p\} \to \mathbb{P}^1$ via projection from p can be extended to an isomorphism on Y. Was our extension unique? We know that, since $\mathbb{P}^1$ is a variety, any two extensions will agree on a closed subset. But they already agreed on a subset of Y whose closure is Y. Hence in this case the extension is unique. We could extend it because Y is a curve, and we can in general extend maps on such subsets curves.

Now consider the map $\phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ defined by $\phi([x : y : z]) = [yz : xz : xy]$. It is well defined on the open subset $\mathbb{P}^2 \setminus \{[0 : 0 : 1], [0 : 1 : 0], [1 : 0 : 0]\}$. This function cannot be extended to all of $\mathbb{P}^2$. In the previous example, we took the limiting process to our projection, the tangent direction at p, to extend the map π_p to all of Y. In this example, there are 2 dimensions worth of directions to approach the holes in the domain, so only a very special function would be extendable. An extension, if it existed, would still be unique by our previous discussion.

Definition 22. *A rational map of prevarieties, $f : Y \dashrightarrow X$, is an equivalence class of maps $(U, f_U : U \dashrightarrow X)$, U open, f a morphism, where two maps are equivalent if they agree on a non-empty open subset of the intersection of their domains.*

Remark 15.3. *If $X = \mathbb{A}^1 = k$, then the collection of rational maps $f : Y \dashrightarrow \mathbb{A}^1$ is $k(Y)$.*

Definition 23. *A rational map $f : Y \dashrightarrow X$ is dominant if there exists a representative for its class with image dense in X.*

Example 39. *Consider $f : \mathbb{A}^2 \dashrightarrow \mathbb{A}^1$ defined by $f(x, y) = y/x$. It is, of course, only defined on $\mathbb{A}^2_x$. The map f is surjective, hence dominant.*

Theorem 15.4. *If X and Y are varieties, there exists a* 1-1 *correspondence between dominant rational maps, $\phi : Y \dashrightarrow X$ and k-algebra morphisms of function fields $\phi^* : A(X) \to A(Y)$.*

Proof. Suppose $\phi : Y \dashrightarrow X$ is dominant, where ϕ is defined on $U \subseteq X$ open. Then $\phi^* : k(X) \to k(Y)$ can be defined as $\phi^*(\langle V, \eta\rangle) = \langle \phi^{-1}(V), \eta \circ \phi\rangle$. Note that $\phi^{-1}(V)$ is not empty because the image of ϕ is dense. It is left as an exercise to see that ϕ is well-defined and a k-algebra morphism.

Conversely, let $\theta : k(X) \to k(Y)$ be a k-algebra morphism of fields. As it cannot be trivial, it must be injective. Take $\emptyset \neq U \subset X$ open affine, where we think of $U \subseteq \mathbb{A}^n$. Then $O_X(U) = A(U)$, and the coordinate functions of $\mathbb{A}^n$, $x^1, \ldots, x^n \in O_X(U)$, are a finite set of generators for $O_X(U)$ as a k-algebra. Then $\theta(x^1), \ldots, \theta(x^n) \in k(Y)$. There exists an open dense subset $V \subset Y$, which we may assume to be affine, on which they are all defined. Now we can think of $\theta : A(U) \to A(V)$. From our previous work, we know that this is equivalent to a morphism $f : V \to U$ such that $f^* = \theta$. Then for $y \in V$, $f(y) = (\theta(x^1)(y), \ldots, \theta(x^n)(y))$. Then $f : Y \dashrightarrow X$ is a rational map. Suppose the image of f has closure $Z \subsetneq X$. Then $0 \subsetneq I(Z)$ is in the kernel of θ, contradicting its injectivity. It's easy to check that these two processes are inverses, so we have established a bijection. □

Definition 24. *A dominant rational map $f : Y \dashrightarrow X$ is birational if and only if there exists a dominant map $g : X \dashrightarrow Y$ such that $f \circ g = id_X$ and $g \circ f = id_Y$ as rational maps. In this case we say that X and Y are birationally isomorphic.*

Proposition 15.5. *Let X and Y be varieties. Then the following are equivalent:*

(1) X and Y are birationally isomorphic,

(2) $k(X) \simeq k(Y)$ as k-algebras, and

(3) there exists a non-empty open sets $U \subseteq X$ and $V \subseteq Y$ such that U and V are isomorphic as varieties.

We have essentially already proven this theorem.

Definition 25. *If X is a variety, we say that X is rational if it is birational to $\mathbb{P}^n$ (or $\mathbb{A}^n$), or equivalently, if $k(X) \simeq k(x_1, \ldots, x_n)$ for some n.*

Example 40. *Let $\phi : \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ be the Segre embedding, and call the quadric surface, image of ϕ, Q. Then Q is rational but not isomorphic to $\mathbb{P}^2$. Consider the copy of $\mathbb{A}^3$ sitting inside $\mathbb{P}^3$, with fourth homogeneous coordinate 1. Here, the equation for Q is $xy = z$. Then $k(Q) = frac(k[x, y, z]/(xy - z)) = frac(k[x, y]) = k(x, y)$. However, $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$ are not isomorphic. We have previously covered Q with disjoint rulings by lines. Suppose C and D are two curves in $\mathbb{P}^2$. Then $\mathbb{P}^2 \setminus C$ is affine, which we can see using the Veronese map from the homework. But D is projective and hence cannot sit in an affine subset, so D is not contained in the compliment of C, so they intersect.*

16 2/28 - Ghosh

Historical Intermezzo:
The Italian School consisting of Cremona,Severi,Enriques,Segre and others dealt with $X \subset \mathbb{P}^n$. They defined dimension and degree in the following manner.

Definition 26. *$dimX = d$, if $H_1, \dots, H_n$ are general hyperplanes in $\mathbb{P}^n$ then $\#(X \bigcap H_1 \cdots \bigcap H_n) < \infty$.*

Definition 27. *$degX$ is defined as $\#(X \bigcap H_1 \cdots \bigcap H_n)$.*

However, these definitions only work for projective varieties.

Definition 28. *Let X be a prevariety over k then $dimX = tr.deg_k k(X)$.*
Recall that $tr.deg_k k(X) = n$ iff $\exists f_1, \dots, f_n \in k(X)$ which are algebrically independent over k and $k(f_1, \dots, f_n) \hookrightarrow k(x_1, \dots x_n)$ is an algebraic extension.

Remark 16.1. *1. Loosely speaking it the $dimX$ is the maximum number of linear coordinates on X.*
2. $\emptyset \neq U \subseteq X$, $k(U) = k(X) \Rightarrow dimX = dimU$.
3. $dim\mathcal{A}^n = n$
4. dim $X = 0$ iff $k(X) = k$ iff X is a point.

Proposition 16.2. *If X is a prevariety , $Y \subsetneq X$ is a proper subvariety $\Rightarrow dimY < dimX$*

Proof. $Y \subsetneq X$. Pick $U \subseteq X$ affine and $U \bigcap Y \neq \emptyset$.
$dimX = dimU$
$dimY = dim(Y \bigcap U)$
So we restrict ourselves to the affine case . Let $R = \mathcal{O}_X(U)$. Then $Y \bigcap U$ is an affine subvariety given by some ideal $\mathfrak{F} = \mathcal{I}(Y \bigcap U) \subseteq R$ and $\mathfrak{F}$ is prime.

$$dim(Y \bigcap U) = tr.deg_k Q(R/\mathfrak{F}) = tr.deg_k(R/\mathfrak{F})$$

We have to prove $tr.deg_k(R/\mathfrak{F}) < tr.deg_k(R)$

Lemma 16.3. *Let R be an integral domain, $(0) \neq \mathfrak{F} \subseteq R$ prime . Then if $tr.deg_k(R) < \infty$, then $tr.deg_k(R/\mathfrak{F}) < tr.deg_k(R)$*

Proof. Suppose $tr.deg_k(R) = n$. Suppose $\exists\, x_1, \dots, x_n \in R$ such that $\overline{x_1} = x_1 \bmod \mathfrak{F}, \dots, \overline{x_n} = x_n \bmod \mathfrak{F}$.

Fix $q \in \mathfrak{F} - 0$.
$\therefore q, x_1, \dots, x_n \in R \Rightarrow \exists\, f \in k[y_0, \dots, y_n]$ such that $f(q, x_1, \dots, x_n) = 0$.
We may also assume f is irreducible . Also f cannot be y_0.
Hence $F(y_1, \dots, y_n) = f(0, y_1, \dots, y_n) \neq 0$.
$F(\overline{x_1}, \dots, \overline{x_n}) = f(\bar{q}, \bar{x_1}, \dots, \bar{x_n}) = f(q, x_1, \dots, x_n) = 0$.
Hence $\overline{x_1}, \dots, \overline{x_n}$ are not algebrically independent. A contradiction.

Definition 29. *$Y \subseteq X$ be a subvariety, then $codim(Y, X) = dimX - dimY \geq 0$*

Theorem 16.4. *Let X be a variety and $U \subseteq X$ be an open set . Let $g \in \mathcal{O}_X(U)$ and Z an irreducible component of $\mathcal{Z}(g) = \{x \in U : g(x) = 0\}$. Then $dimZ = dimX - 1$.*

Example 41. *$Q : xz = yw$ in $\mathbb{P}^3$*
$f = \frac{x^2+z^2-3xw}{y^2+yx+3z^2}$
Any component of the locus $Q \bigcap \mathcal{Z}(f)$ has dimension 1.

Proof. Let $U_0 \subseteq U$ be affine such that $U_0 \bigcap \mathcal{Z}(g) \neq \emptyset$.
$R = \mathcal{O}_X(U_0)$
$f := g|_{U_0}$ corresponds to the ideal (f) .
The irreducible components of $U_0 \bigcap \mathcal{Z}(g)$ correspond to minimal primes $\mathfrak{F} \subseteq R$ such that $(f) \subseteq \mathfrak{F}$. We want $tr.deg_k(R/\mathfrak{F}) \leq tr.deg_k(R) - 1$

Theorem 16.5. (**KrullHaupidealsatz**) *Let R be a finitely generated integral domain over k, let $0 \neq f \in \mathfrak{F}$ be a minimal prime ideal. Then*
$tr.deg_k(R/\mathfrak{F}) \leq tr.deg_k(R) - 1$

Corollary 16.6. *If X is a variety and $Z \subseteq X$ is a maximal closed proper subset . Then* $dimZ = dimX - 1$

Proof. Suppose $dimZ \leq dimX - 2$. We can assume X is affine .
$R = \mathcal{O}_X(X)$ and $(0) \neq I(Z) \subseteq R$ a prime ideal.
Take $f \in \mathcal{I}(Z) \Rightarrow Z \subsetneq \mathcal{Z}(f) \subset X$
$Z \neq Z(f)$ because each component of $\mathcal{Z}(f)$ has codimension 1.
This contradicts the maximality of Z.

Corollary 16.7. $dimX = max\{r : \exists \emptyset \neq Z_0 \subset Z_1 \subset \dots Z_r = X\}$. *The Z_i's are closed irreducible subsets*

Proof. Induction on r.

A third approach to dimension

Let R be a localring . Let $\mathcal{M} \subseteq R$ be a maximal ideal.
$dim(R) = max\{n, \exists (0) = \mathfrak{F}_0 \subset \mathfrak{F}_1 \cdots \subset \mathfrak{F}_n = \mathcal{M}\}$. The chain is a chain of prime ideals. This is the Krull dimension of R.

Proposition 16.8. *If X is a variety and $p \in X \Rightarrow dimX = dim\mathcal{O}_{X,p}$*

Proof. Let X be affine and $p \in X$ and $R = \mathcal{O}_X(X)$
$\mathcal{O}_{X,p} = R_{\mathcal{M}_p}$ and $\mathcal{M} = m_p$
There is a one to one correspondence between prime ideals in $R_{\mathcal{M}}$ and prime ideals of R contained in $\mathcal{M}$.

Chain of prime ideals contained in $\mathcal{M}_p \leftrightarrow$ chain of irreducible closed subsets starting with $Z_0 = \{p\}$.
$\therefore dim\mathcal{O}_{X,p} = dimX$.

17 3/2 - Charters

Recall from the previous class the result that if X is an affine variety and f is a non-zero function, then each component of $Z(f)$ has dimension dim $X - 1$. That is, $Z(f)$ has pure codimension 1.

Proposition 17.1. *If X is an affine variety and $f_1, \dots, f_t \in \mathcal{O}_X(X)$, then each component of $Z(f_1, \dots, f_t)$ has dimension $>$ dim $X - t$. (That is, the codimension of each componenet is $\leq t$).*

Proof. Induction on t. □

Remark 17.2. *There are lots of examples when dim* $Z(f_1, \ldots, f_t) > \dim X - t$. *For example, polynomials that were independent in a larger ring could become dependent in the quotient ring.*

Example 42. *Recall the twisted cubic in* $\mathbb{P}^3$ *described by*

$$\gamma : \mathbb{P}^1 \to \mathbb{P}^3 \ : \ \gamma[s,t] = [s^3, s^2t, st^2, t^3] = [x_0, x_1, x_2, x_3].$$

Then

$$\Gamma = Im(\gamma) = \left\{ [x_0, x_1, x_2, x_3] \in \mathbb{P}^3 \ : \ det_2 \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix} = 0 \right\}$$

Thus we can see that three quadrics (the three 2*-dimensinal determinants) will cut out the (1-dimensional) twisted cubic.*

Does the converse of our theorem hold?

Theorem 17.3. *Let* X *be an affine variety,* Z *a subvariety of* X *(closed, irreducible) of codimension* t. *Then there exist a finite number of functions* $f_1, \ldots, f_t \in \mathcal{O}_X(X)$ *such that* Z *is a component of* $Z(f_1, \ldots, f_t)$.

Proof. First let $t = 1$. Then $Z \subset X$ has codimension 1. Let $R = \mathcal{O}_X(X)$. Then $0 \neq I(Z) \subseteq R$ is a prime ideal of R. Choose $0 \neq f \in I(Z)$. Since f vanishes on Z, $Z \subseteq Z(f)$. But now Z has codimension 1, and $Z(f)$ has pure codimension 1 by Krull's theorem. It follows by a dimension argument that we must have Z as a component of $Z(f)$.

Now suppose that $t \geq 2$. Let $0 \neq f \in I(Z)$. Then $Z(f) = Z_1 \cup \cdots \cup Z_r$, Z_i the irreducible componenets of Z. Each of these components has codimension 1 by assumption. We know that $\text{codim}(Z, X) \geq 2$. It follows that we have $Z \subseteq Z(f)$, but $Z_i \nsubseteq Z$ by a dimension argument. In particular, $I(Z) \nsubseteq I(Z_i)$ for any i. Hence there exists some $g \in I(Z) \setminus I(Z_1) \cup \cdots \cup I(Z_r)$, as the $I(Z_i)$ are prime ideals. Thus we have found a function g that vanishes on Z but not on Z_i for any i. Each component of the zero locus $Z(f, g)$ is of pure codimension 2, however:

$$Z \subseteq Z(f, g) = (Z_1 \cap \{g = 0\}) \cup \cdots \cup (Z_r \cap \{g = 0\})$$

where $Z_i \cap \{g = 0\} \subsetneq Z_i$. Iterate this process to find functions $f_1, \ldots, f_t \in I(Z)$ such that $Z(f_1, \ldots, f_t)$ is of pure codimension t and $Z \subseteq Z(f_1, \ldots, f_t)$. Hence Z is one of the irreducible components of $Z(f_1, \ldots, f_t)$. □

Definition 30. *Let* X *be a variety. Then* $Z \subseteq X$, $codim(Z, X) = t$ *is a local complete intersection if for all* $p \in Z$, *there exists some affine open neighborhood* $U \subseteq X$ *containing* p *and* $f_1, \ldots, f_t \in \mathcal{O}_X(X)$ *such that* $Z \cap U = \{x \in U \ : \ f_1(x) = \cdots = f_t(x) = 0\}$.

[Note: we proved before that Z was a component of this locus.]

Remark 17.4. *Even in codimension* 1 *there are subvarieties which are not local complete intersections.*

Theorem 17.5. *If* X *is an affine variety such that* $A(X) = \mathcal{O}_X(X)$ *is a Unique Factorization Domain, then every codimension* 1 *subvariety of* X *is of type* $Z(f)$ *for some* $f \in \mathcal{O}_X(X)$.

Remark 17.6. *Let* $X = \mathbb{A}^n$. *Then* $A(X) = k[x_1, \ldots, x_n]$ *is a UFD. Hence every codimension* 1 *subvariety is a hypersurface.*

Example 43. $X = \mathbb{A}^3$, $C = \{(t^3, t^4, t^5) \;:\; t \in \mathbb{C}\}$. *Then C cannot be cut out by two equations.*

Proof. Suppose $R = \mathcal{O}_X(X)$ is a UFD. Then every nonzero minimal prime ideal is principal. To see that this is true, suppose $0 \neq \mathfrak{P} \subseteq R$ is a minimal prime ideal of R. Let $0 \neq f \in \mathfrak{P}$. Take f' to be any prime factor of f such that $f' \in \mathfrak{P}$. Then (f') is a prime ideal of R contained in $\mathfrak{P}$. Then it follows by the minimality of $\mathfrak{P}$ that $(f') = \mathfrak{P}$, so $\mathfrak{P}$ is principal as desired.

Now let $Z \subset X$ be of pure codimension 1, with $Z = Z_1 \cup \cdots \cup Z_r$, where each Z_i is of codimension 1 in X. The Z_i are maximal irreducible sets in X, and thus correspond in a 1-1 manner with the minimal prime ideals in $A(X)$, which recall are all principal. It follows that $I(Z_i) = (f_i)$, and thus $I(Z) = (f_1 f_2 \cdots f_r)$, a principal ideal as desired. □

18 Smooth Varieties

To get a feel for smooth varieties, think about which varieties we want to consider smooth. Consider the varieties

$$X = \{x^2 + y^2 = 1\},\; Y = \{y^2 = x^3 + x^2\},\; Z = \{y^2 = x^3\}.$$

The first of these is a circle, which clearly has a neighborhood of each $p \in X$ which is isomorphic to $\mathbb{C}$. Thus we want X to be smooth. Y, however, has a point where the function is not 1-1 - i.e., the image of the function in affine space crosses itself. Thus this function is not smooth. Finally, Z has a cusp - a point where it is not differentiable as we think of it in standard calculus. We want Z to also not be smooth.

Definition 31. *Let $X \subseteq \mathbb{A}^n$ be an affine variety and let $p \in X$. Then $I(X) = \langle f_1, \ldots, f_t \rangle$, where the f_i are generators of $I(X)$. Define the Jacobian matrix at p by*

$$J_p = \left(\frac{\partial f_i}{\partial x_j}(p) \right), i = 1, \ldots, t; j = 1, \ldots, n.$$

We say that p is a smooth point if Rank$(J(p)) = n - d$, where $d = \dim X$.

Remark 18.1. *We always have $t \geq n - d$. But it is not true that you can always pick $n - d$ generators.*

Example 44. *Let*

$$X^1 = \left\{ \begin{array}{c} x + y + z = 0 \\ x^2 + 2y + z^3 = 0 \end{array} \right. \subseteq \mathbb{C}^3.$$

Let $p = (0, 0, 0)$. Then

$$J_0(X) = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \end{pmatrix}$$

It is clear that Rank$(J(X)_0) = 2$, hence X is smooth at p. [I.e., X is locally isomorphic to $\mathbb{C}$.] To see the local isomorphism, define a map from $X \to \mathbb{C}$ by

$$(x, y, z) \mapsto x.$$

This function is a local isomorphism. The fact that this is so is given by the implicit function theorem, which tells us that given $f_1(x, y, z) = 0$, $f_2(x, y, z) = 0$, we can plug in $y = y(x)$, $z = z(x)$ and solve about the point $(0, 0, 0)$.

Definition 32. *If R is a Noetherian local ring over k with maximal ideal $\mathfrak{M}$, we say that R is regular if dim $\mathfrak{M}/\mathfrak{M}^2 =$ dim R, the Krull dimension of R.*

Example 45. *Note that we always have dim $\mathfrak{M}/\mathfrak{M}^2 \geq$ dim R.*

Theorem 18.2. *Let $X \subseteq \mathbb{A}^n$ be an affine variety. Then X is smooth at the point $p \in X$ if and only if $\mathcal{O}_{X,p}$ is a regular local ring. I.e. if and only if we have dim $\mathfrak{M}_p/\mathfrak{M}_p^2 =$ dim $\mathcal{O}_{X,p} =$ dim X.*

Definition 33. *If $(X, \mathcal{O}_X)$ is an abstract variety, and $p \in X$ then X is smooth at p if and only if $\mathcal{O}_{X,p}$ is regular. I.e., dim $\mathfrak{M}_p/\mathfrak{M}_p^2 =$ dim X.*

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20 Smooth Varieties (cont.)

Recall our setup from last time: let $X^d \subseteq \mathbb{A}^n$ be an affine variety with $I(X) = (f_1, \ldots, f_t)$. We say $p \in X$ is a *smooth* (or *nonsingular* or *regular*) point, which we denote $p \in X_{\text{reg}}$, if $\operatorname{rank}\left(\frac{\partial f_\alpha}{\partial x_j}\right)_{\substack{1\le\alpha\le t\\ 1\le j\le n}} = n - d$. (We take the opportunity here to note that this is *formal* differentiation of polynomials.) Algebraic geometry since Grothendieck has been about developing invariant concepts that work in a wide variety of cases. We want concepts that work for general algebraic varieties and also (later) for schemes. When we develop schemes, we will tackle deformation problems about how varieties vary in families. So let us make a more general definition:

Definition 34. *We say a variety X is* smooth *at a point $p \in X$ if $\mathcal{O}_{X,p}$ is a regular local ring. (Recall $\mathcal{O}_{X,p}$ is a regular local ring if $\dim_k \mathfrak{m}_p/{\mathfrak{m}_p}^2 = \dim \mathcal{O}_{X,p} = \dim X$).*

This is a definition that does not depend on the embedding of X or the choice of generators. We now show that these two definitions are, in fact, equivalent in the affine case:

Proposition 20.1. *Let $X^d \subseteq \mathbb{A}^n$ as above. Then* $\operatorname{rank}\left(\frac{\partial f_\alpha}{\partial x_j}\right)_{\substack{1\le\alpha\le t\\ 1\le j\le n}} = n - d$ *if and only if $\mathcal{O}_{X,p}$ is a regular local ring.*

Proof. Let $p = (a_1, \ldots, a_n) \in X$, and denote by $\mathfrak{a}_p = (x_1 - a_1, \ldots, x_n - a_n) \subseteq k[x_1, \ldots, x_n]$ the maximal ideal corresponding to p in $\mathbb{A}^n$. We wish to identify $\mathfrak{a}_p/{\mathfrak{a}_p}^2$ with the vector space of hyperplanes in $\mathbb{A}^n$ passing through the origin. Define the map:

$$\theta : \mathfrak{a}_p/{\mathfrak{a}_p}^2 \xrightarrow{\sim} k^n$$

$$\theta(f) = \left\langle \frac{\partial f}{\partial x_1}(p), \ldots, \frac{\partial f}{\partial x_n}(p) \right\rangle.$$

Again we note that this is formal differentiation of polynomials, and all of the old rules apply. In particular Leibnitz's rule tells us that for $f, g \in \mathfrak{a}_p$, we have $\theta(fg) = 0$. So for each f we have an associated hyperplane though 0 given by $\sum \frac{\partial f}{\partial x_i}(p) x_i = 0$. Note that the $\theta(x_i)$ form a basis for k^n. So let $I(X) \subseteq R = k[x_1, \ldots, x_n]$ and $I(X) = (f_1, \ldots, f_r)$, then we claim that

$\operatorname{rank} J = \dim \theta(I(X))$ where $J = \left(\dfrac{\partial f_\alpha}{\partial x_j}\right)_{\substack{1\le\alpha\le t \\ 1\le j\le n}}$ is the Jacobian. To see this, note that $\theta(f_i)$ span $\theta(I(X))$, and that $\dim \theta(I(X))$ is indeed the rank of the matrix formed by the $\theta(f_i)$, which is J. Then we have the isomorphism:

$$\begin{aligned}\theta(I(X)) &\cong I(X)/(I(X) \cap \mathfrak{a}_p{}^2) \\ &\cong \frac{I(X) + \mathfrak{a}_p{}^2}{\mathfrak{a}_p{}^2}\end{aligned}$$

Recall that $\mathcal{O}_{X,p} = (R/I(X))_{\widetilde{\mathfrak{a}}_p}$ where $\widetilde{\mathfrak{a}}_p$ is the maximal ideal in the quotient corresponding to $\mathfrak{a}_p$. Now $\mathfrak{m}_p \subseteq \mathcal{O}_{X,p}$. Note that $R_\mathfrak{m}$ is of course a local ring and $\widetilde{\mathfrak{m}} = \mathfrak{m}R \subset R_\mathfrak{m}$ is its maximal ideal, and we have a natural isomorphism $\widetilde{\mathfrak{m}}/\widetilde{\mathfrak{m}}^2 \cong \mathfrak{m}/\mathfrak{m}^2$. So

$$\mathfrak{m}_p/\mathfrak{m}_p{}^2 \cong \widetilde{\mathfrak{a}}_p/\widetilde{\mathfrak{a}}_p^2 \cong \frac{\mathfrak{a}_p + I(X)}{\mathfrak{a}_p{}^2 + I(X)} \cong \frac{\mathfrak{a}_p}{\mathfrak{a}_p{}^2 + I(X)}$$

since $I(X) \subseteq \mathfrak{a}_p$. Now note that we have

$$\mathfrak{a}_p{}^2 \subseteq \mathfrak{a}_p{}^2 + I(X) \subseteq \mathfrak{a}_p$$

so

$$\dim \frac{\mathfrak{a}_p}{\mathfrak{a}_p{}^2 + I(X)} + \dim \frac{\mathfrak{a}_p{}^2 + I(X)}{\mathfrak{a}_p{}^2} = \dim \mathfrak{a}_p/\mathfrak{a}_p{}^2 = n$$

since we showed that $\theta : \mathfrak{a}_p/\mathfrak{a}_p{}^2 \xrightarrow{\sim} k^n$ and $\dim k^n = n$. So we have

$$\dim \mathfrak{m}_p/\mathfrak{m}_p{}^2 + \operatorname{rank}(J) = n$$

and therefore

$$\dim \mathfrak{m}_p/\mathfrak{m}_p{}^2 = d \iff \operatorname{rank}(J) = n - d$$

so our two definitions are in fact equivalent. ☐

21 The Zariski Tangent Space

Definition 35. *Let $p \in X$. Then we define the* Zariski tangent space $T_p(X)$ *to be*

$$T_p(X) = \left(\mathfrak{m}_p/\mathfrak{m}_p{}^2\right)^*.$$

What we proved above was that X is smooth at p iff $\dim T_p(X) = \dim X$. Note that in general $\dim \mathfrak{m}_p/\mathfrak{m}_p{}^2 \geq \dim X$ with equality iff p is a smooth point of X, so in particular $\dim T_p(X) \geq \dim(X)$.

Remark 21.1. *The set $X_{sing} = \{x \in X : X \text{ is smooth at } x\}^c$ (note the complement) is a proper Zariski closed subset in X, that is, $X_{reg} = \{x \in X : X \text{ is smooth at } x\}$ is an open dense subset.*

Since the singularity of a point is a local matter, we might as well assume X is affine. Let $0 = p \in X \subseteq \mathbb{A}^n$, $I(X) \subseteq k[x_1, \dots, x_n]$. Define

$$k[x_1, \dots, x_n]^1 = \{a_1x_1 + \cdots + a_nx_n : a_i \in k\}.$$

We can always decompose $f = f^{(0)} + f^{(1)} + \cdots + f^{(d)}$ where $d = \deg f$ and the $f^{(i)}$ are homogenous of degree i. Then we define

$$I(X)^1 = \{f^{(1)} : f \in I(X)\}$$

to the be linear part of the ideal.

Proposition 21.2. *There exists an isomorphism*

$$\eta : \mathfrak{m}_p/\mathfrak{m}_p{}^2 \xrightarrow{\sim} k[x_1, \dots, x_n]^1/I(X)^1$$

so $T_p(X) \cong Z(I(X)^1) = \{x \in \mathbb{A}^n : f^1(x) = 0 \ \forall f \in I(X)\}$. *Since we can write*

$$f = \underbrace{\sum_i \frac{\partial f}{\partial x_i}(p)x_i}_{f^{(1)}} + f^{(2)} + \cdots + f^{(d)},$$

we can write the tangent space as

$$T_p(X) = \{x \in \mathbb{A}^n : \sum_i \frac{\partial f}{\partial x_i}(p)x_i = 0 \ \forall f \in I(X)\}.$$

Proof. Let $\phi \in \mathfrak{m}_p$. Define

$$\eta(\phi) = \sum_{i=1}^n \frac{\partial \phi}{\partial x_i}(p)x_i \bmod I(X).$$

Recall that $\phi = \dfrac{f(x_1, \dots, x_n)}{g(x_1, \dots, x_n)}$ for some $f, g \in A(X)$, $g(p) \neq 0$. Note that f, g are not uniquely determined as polynomials in $k[x_1, \dots, x_n]$, in fact they are only defined up to $f + I(X)$. This is why we must mod out by $I(X)$ to make η well-defined. Note that

$$\eta^{-1}\left(\sum a_ix_i\right) = \sum a_ix_i \bmod \mathfrak{m}_p{}^2$$

so η is indeed the desired isomorphism, and the result for the tangent space follows. $\square$

Let's do some examples.

Example 46. *Consider the curve* $X : y = x^2 - x$ *in* $\mathbb{A}^2$, *and let* $p = (0, 0)$. *Then* $T_p(X) = \{y + x = 0\}$, *since* $f(x, y) = y - x^2 + x$, *we have* $f^{(1)} = y + x$. *Note in particular that* $\dim T_pX = 1$ *so* $p = (0, 0)$ *and* $p \in X_{reg}$.

Example 47. *Consider the nodal curve* $X : y^2 = x^2 + x^3$, *and let* $p = (0, 0)$. *Then* $f(x, y) = y^2 - x^2 - x^3$, *and we have* $f^{(1)} = 0$, *so* $T_p(X) = \mathbb{A}^2$. *Therefore* X *is singular at* p, *but* X *is smooth everywhere else.*

Example 48. *Consider the cuspidal curve $X : y^2 = x^3$, and let $p = (0,0)$. Then $f(x,y) = y^2 - x^3$, and we have $f^{(1)} = 0$, so again $T_p(X) = \mathbb{A}^2$, and X is singular at p.*

There is a whole branch of algebraic geometry devoted to singularity theory. The simplest possible singularity is a node.

Example 49. *A marginally worse singularity is given in the tacnode curve $X : x^2 = x^4 + y^4$. Let $p = (0,0)$, and note once again that $T_p(X) = \mathbb{A}^2$, and X is singular at p.*

Example 50. *Consider now the Fermat hypersurface, $X = \{[x_0, \ldots, x_n] \in \mathbb{P}^n : {x_0}^d + \cdots + {x_n}^d =\}$. Then $\dfrac{\partial f}{\partial x_i} = d\, {x_i}^{d-1}$. Suppose now that the characteristic of our underlying field $\operatorname{char} k = d$ (is a prime). Then all of the partials vanish and the Fermat hypersurface is singular.* ***Caution:*** *Keep this in mind, and remember that $\frac{d}{dx} x^p = 0$ in characteristic p.*

Remark 21.3. *There is yet another description of the Zariski tangent space. We can define $T_p(X)$ to be the vector space of derivations $D : \mathcal{O}_{X,p} \to k$, that is, the space of k-linear functions that satisfy the Leibnitz rule, $D(fg) = f(p)D(g) + g(p)D(f)$. To see this, note that $T_p(X) = Hom_k(\mathfrak{m}_p/{\mathfrak{m}_p}^2, k)$. So consider the map $(\ell : \mathfrak{m}_p/{\mathfrak{m}_p}^2 \to k) \rightsquigarrow D_\ell(f)$ where $D_\ell(f) = \ell(f - f(p))$. We must check that the Leibnitz rule is satisfied by D_ℓ, but that easily follows since ℓ annihilates the square, ${\mathfrak{m}_p}^2$, as we remarked above.*

21.1 Smooth Projective Varieties

It is useful to have a projective Jacobian criterion. Suppose $X \subseteq \mathbb{P}^n$ is a projective variety, and $p \in X$. Let $I(X) = (f_1, \ldots, f_t)$ be the homogenous ideal in $n+1$ variables. Then X is smooth at p if and only if

$$\operatorname{rank}(J) = \operatorname{rank}\left(\frac{\partial f_\alpha}{\partial x_i}(p)\right)_{\substack{\alpha=1,\ldots,t \\ i=0,\ldots,n}} = n - \dim X.$$

This is just an application of the Euler rule for homogenous polynomials, $\sum x_i \dfrac{\partial F}{\partial x_i} = d \cdot F$ where $d = \deg F$, which implies that if $F(p) = 0$ for some $p \in X$ then $\sum x_i \dfrac{\partial F}{\partial x_i} = 0$ as well at p.

Example 51. *We will show that the twisted cubic $X \subseteq \mathbb{P}^3$, which we recall is given by the vanishing of the 2×2 minors of the following matrix:*

$$X : \det{}_2 \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix} = 0,$$

which is the vanishing of the quadratics

$$\begin{aligned} Q_1 &= x_0x_2 - {x_1}^2 \\ Q_2 &= x_0x_3 - x_1x_2 \\ Q_3 &= x_1x_3 - {x_2}^2 \end{aligned}$$

Then we have

$$J = \begin{pmatrix} x_2 & -2x_1 & x_0 & 0 \\ x_3 & -x_2 & -x_1 & x_0 \\ 0 & x_3 & -2x_2 & x_1 \end{pmatrix}.$$

We claim that $\mathrm{rank}(J) = 2$. *If* J *had rank 1 then we could check the* 2×2 *minors of* J, *for example*

$$\begin{vmatrix} x_3 & -x_2 \\ 0 & x_3 \end{vmatrix} = 0 \implies x_3 = 0$$

and similarly we conclude that $x_1 = 0$, *and then we must have* $x_0 = x_2 = 0$, *but this is impossible. Therefore the twisted cubic is indeed smooth.*

21.2 Differentials of Morphisms

Suppose $f : X \to Y$ is a morphism. Then there is a k-linear map:

$$df_p = T_p(f) : T_p(X) \to T_{f(p)}Y$$

where we recall that $T_p(X) = (\mathfrak{m}_p/{\mathfrak{m}_p}^2)^*$ and $T_{f(p)} = (\mathfrak{m}_{f(p)}/{\mathfrak{m}_{f(p)}}^2)^*$. To see this, recall that we have an induced map of the local rings

$$f_p^{\#} : \mathcal{O}_{Y,f(p)} \to \mathcal{O}_{X,p}$$

given by the pullback of functions, and therefore

$$f_p^{\#}(\mathfrak{m}_{f(p)}) \subseteq \mathfrak{m}_p \qquad \text{and} \qquad f_p^{\#}({\mathfrak{m}_{f(p)}}^2) \subseteq {\mathfrak{m}_p}^2$$

so we have an induced map:

$$f_p^{\#} : \mathfrak{m}_{f(p)}/{\mathfrak{m}_{f(p)}}^2 \to \mathfrak{m}_p/{\mathfrak{m}_p}^2$$

and then the dual of this map gives us $df_p = T_p(f) = (f_p^{\#})^*$, the desired map.

How do we describe the tangent map locally? Suppose we have a map $f : X \to Y$ where $X \subseteq \mathbb{A}^n$, $Y \subseteq \mathbb{A}^m$ and $f = (f_1, \ldots, f_m)$. Assume for simplicity that $p = 0$ is the origin. Then we showed that the $x_1, \ldots x_n$ span our space of hyperplanes through the origin in $\mathbb{A}^n$, so we write this as vector space as $k\langle x_1, \ldots, x_n \rangle$, and $T_p(X) \subseteq T_p(\mathbb{A}^n)$ and we write

$$T_p(\mathbb{A}^n) = (k\langle x_1, \ldots, x_n \rangle)^* = k \left\langle \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \right\rangle$$

where

$$\frac{\partial}{\partial x_i} : k^n \to k$$

is the dual to x_i. Then

$$T_p(X) = \left\{ v = \sum a_i \frac{\partial}{\partial x_i} : \forall f \in I(X),\ \sum a_i \frac{\partial f}{\partial x_i}(p) = 0 \right\}$$

and we have the map

$$\begin{array}{c} T_p(X) \subseteq k\left\langle \dfrac{\partial}{\partial x_1}, \ldots, \dfrac{\partial}{\partial x_n} \right\rangle \\ {\scriptstyle T_p(f)}\Big\downarrow \\ T_p(Y) \subseteq k\left\langle \dfrac{\partial}{\partial y_1}, \ldots, \dfrac{\partial}{\partial y_m} \right\rangle \end{array}$$

$T_p(f)$ is the linear map with matrix

$$\left(\frac{\partial f_\alpha}{\partial x_i}(p) \right)_{\substack{i=1,\ldots,n \\ \alpha=1,\ldots,m}}.$$

This is how we obtain the tangent map, just as we did in differential geometry.

In calculus, we had an inverse function theorem when the differential was an isomorphism. *The inverse function theorem fails in an algebraic setting,* which isn't really a surprise. Even if we had a morphism $f : X \to Y$ and $p \in X$ such that $T_p(f) : T_p(X) \xrightarrow{\sim} T_p(Y)$, to require that an open neighborhood be isomorphic would be a *very* strong condition, in fact it would make X and Y birationally isomorphic. This is much different than the situation in analysis.

Example 52. *Let $X : y = x^2 \subset \mathbb{A}^2$. Let $\pi : X \to \mathbb{A}^1$ be the projection of the y coordinate. Then we have the map*

$$T_{(x,x^2)}(\pi) : T_{(x,x^2)}(X) \to T_{x^2}(\mathbb{A}^1)$$
$$\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right) \mapsto \frac{\partial}{\partial y}$$

which is an isomorphism for all $x \neq 0$. Take for example $p = (1,1)$. Then $T_p(\pi)$ is an ismorphism. In analysis this tells us that $x = \sqrt{y}$, $(x,y) \mapsto y$ is an isomorphism. We claim that $\nexists \varnothing \neq U \subseteq X$ open such that $\pi|_U$ is an isomorphism. Clearly, in the Zariski topology any such U will contain lots of pairs (x,x^2) and $(-x,x^2)$, and therefore the map is not one-to-one.

But it's a still a special condition for $T_p(f)$ to be an isomorphism, and we therefore make the following definition:

Definition 36. *$f : X \to Y$ is* étale *at p if $T_p(f)$ is an isomorphism.*

There is a way to say when the map is étale using local rings.

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Homework review.

22.1 Veronese Embedding

For each d the veronese embedding is defined as follows:

$$\partial_d : \mathbb{P}^m \to \mathbb{P}^N$$

where $N = \begin{pmatrix} n+d \\ d \end{pmatrix} - 1$, is given by

$$\partial_d[x_0 : , \dots , \ : x_m] = [z_0 : , \dots , \ : z_N]$$

with $z_i = x_0^{i_0} x_1^{i_1} \dots x_m^{i_m}, i_0 + \dots + i_m = d$.
In other words $\partial_d[x_0, \dots, x_m] =$ [monomials in x_i of degree d].
Examples.
$m = 1$, $d = 2$.

$$\partial_2 : \mathbb{P}^1 \to \mathbb{P}^2$$
$$\partial_2[x, y] = [x^2 : xy : y^2].$$

The image inside $\mathbb{P}^2$ is a conic.
$m = 1$, $d = 3$.

$$\partial_3 : \mathbb{P}^1 \to \mathbb{P}^3$$
$$\partial_3[x : y] = [x^3 : x^2y : xy^2 : y^3].$$

The image is the projectivized twisted cubic. Recall the twisted cubic is defined by as follows:

Let $[x_0 : x_1 : x_2 : x_3]$ be the coordinates on $\mathbb{P}^3$. Then the twisted cubic is defined as $\det_2 \begin{vmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{vmatrix} = 0$ ($\det_2$ refers to the 2×2 minors). The twisted cubic is cut out by quadrics.

The Veronese surface.

$$\partial_2 : \mathbb{P}^2 \to \mathbb{P}^5$$
$$\partial_2[x, y, z] = [x^2 : y^2 : z^2 : xy : yz : xz].$$

or more generally

$$\partial_2 : \mathbb{P}^n \to \mathbb{P}^{\frac{(n+1)(n+2)}{2}-1}$$

Let $[z_{i,j}] 1 \leq j \leq j \leq n$ be coordinates on $\mathbb{P}^{\frac{(n+1)(n+2)}{2}-1}$.

$\mathfrak{I}(\partial_2(\mathbb{P}^n))$, the ideal of the image, is given by the 2×2 minors of the $(n+1) \times (n+1)$ symmetric matrix with entries $z_{i-1,j-1}$ on the (i, j)th position.

$$A = \begin{vmatrix} z_{00} & z_{01} & z_{02} & \dots & z_{0n} \\ z_{10} & z_{11} & z_{12} & \dots & z_{1n} \\ z_{20} & z_{21} & z_{22} & \dots & z_{2n} \\ \dots & & & \ddots & 0 \\ z_{n0} & & & & z_{nn} \end{vmatrix}$$

In other words $\mathfrak{I}(\partial_2(\mathbb{P}^n))$ is generated by $\det_2(A)$.

Lets go back to the case of the veronese surface in $\mathbb{P}^5$.

$$\partial_2 : \mathbb{P}^2 \to \mathbb{P}^5$$

$$\partial_2[x,y,z] = [x^2 : y^2 : z^2 : xy : yz : xz] = [z_0 : \cdots : z_5].$$

The equation of the surface is given by

$$\det_2 \begin{vmatrix} z_0 & z_1 & z_2 \\ z_1 & z_3 & z_4 \\ z_2 & z_4 & z_{22} \end{vmatrix} = 0.$$

The next question could be to describe explicitely the image $\partial_2(\mathbb{P}^n) = X \subset \mathbb{P}^N$. That is

1. We found generators of the ideal
2. Find the space of all relations with polynomial coefficients between the quadrics (called the sysygies of the quadrics).

Next, let $X_d \subset \mathbb{P}^n$ a hypersurface of degree d. Now map it $\partial_d(X_d) \subset \mathbb{P}^N$ becomes a hyperplane section of $\partial_d(\mathbb{P}^N)$.

That is hypersurfaces of degree d in $\mathbb{P}^n$ become linear sections of $\partial(\mathbb{P}^n) \hookrightarrow \mathbb{P}^N$.

Application.

Theorem 22.1. *Let $X_d \subset \mathbb{P}^n$ be a hypersurface of degree d. Then $\mathbb{P}^n - X_d$ is affine.*

Proof. Use the fact that $\partial_d(\mathbb{P}^n - X_d)$ is a hyperplane complement inside of $\partial_d(\mathbb{P}^n)$. □

This is a very useful technique to prove that something is affine.

22.2 Grassmannians by Rohit Ghosh

$\mathcal{G}(1,n)$. We want to show that $\mathcal{G}(1,n)$ is a projective variety, and we'll do this by showing it is a closed subset of some $\mathbb{P}^N$. Define:

$$\phi : \{\text{Lines in } \mathbb{P}^n\} \to \mathbb{P}^N,\ N = \begin{pmatrix} n+1 \\ 2 \end{pmatrix} - 1$$

Coordinates on $\mathbb{P}^N$, $(p_{0,1}, p_{0,2}, \ldots, p_{n-1,n})$.

Let $\mathcal{L}$ be line in $\mathbb{P}^n$ and let $a = [a_0, \ldots, a_n]$ and $b = [b_0, \ldots b_n]$ be two different points on this line.

Then map is given by

$$A = \begin{pmatrix} a_0 & \ldots & a_n \\ b_0 & \ldots & b_n \end{pmatrix} \to [\det_2(A)]$$

where $\det_2(A)$ denotes the 2×2 minors of A.

We need to check that this map is well defined. Any other two points on $\mathcal{L}$ can be written as $\lambda_1 a + \mu_1 b$ and $\lambda_2 a + \mu_2 b$.

The image of

$$\overline{A} = \begin{pmatrix} \lambda_1 a_0 + \mu_1 b_0 & \ldots & \lambda_1 a_n + \mu_1 b_n \\ \lambda_2 a_0 + \mu_2 b_0 & \ldots & \lambda_2 a_n + \mu_2 b_n \end{pmatrix} \to [\det_2(\overline{A})] = [(\lambda_1\mu_2 - \lambda_2\mu_1)(\det_2(A))] = [\det_2(A)]$$

and the mapping is indeed well defined.

We will see about injectivity later.

If $i_1 \le j_1 \le j_2 \le j_3 \in \{0, \dots n\}$ we claim that the image of of $\mathcal{G}(1,n) \subset \mathbb{P}^N$ is the zero locus of the polynomials

$$-p_{i_1,j_1}p_{j_2,j_3} + p_{i_1,j_2}p_{j_1,j_3} - p_{i_1,j_3}p_{j_1,j_2} = 0$$

There are $\begin{pmatrix} n+1 \\ 4 \end{pmatrix}$ such polynomials.

Lets call the ideal generated by these polynomials $\mathfrak{p}$. It is clear that the image of this map is contained in the zeros of this ideal. So let $p \in Z(\mathfrak{p})$. We can assume $p_{01} = 1$. Then the line in $\mathbb{P}^n$ defined by the two points in $\mathbb{P}^n$, $[1 : 0 : -p_{12} : \dots : -p_{1n}]$ and $[0 : 1 : p_{02} : \dots : p_{0n}]$ maps to this point.

Note that $\mathcal{G}(1,n) \cong G(2, n+1)$ and in general if

$$G(k,n) = \{V \subset k^n : dimV = k\}$$

and this space is canonically isomorphic to $\mathcal{G}(k-1, n-1)$.

What we just saw is a special case of the Plücker embedding.

Define a map $\phi : G(k,n) \to \mathbb{P}(\Lambda^k(V))$ as follows. Let $V \subset k^n$ be a k dimensional subspace. Choose any basis of V say $(v_1, \dots v_k)$. Then

$$\phi(V) = [v_1 \wedge \dots \wedge v_k] \in \mathbb{P}(\Lambda^k(V))$$

It is easy to see that this map is well defined. If we set $w = v_1 \wedge \dots \wedge v_k$ we can recover the subspace V from a point in the image of ϕ as the set of all v such that $v \wedge w = 0$. This says the map ϕ is injective (as was claimed earlier).

Hence, we can identify $G(k,n) \subset \mathbb{P}(\Lambda^k(V))$ as the space of totally decomposable vectors (can be written as $v_1 \wedge \dots \wedge v_k$ for some vectors v_i).

Lets get specific and suppose that $k = 2$.

Now $w \in \Lambda^2(V)$ is totally decomposable if and only if $w \wedge w = 0$. If we choose a basis for V say $\{e_i\}$, then set $p_{ij} = e_i \wedge e_j$ then the equation $w \wedge w = 0$ are exactly the quadratic equations we wrote earlier for describing the image of $\mathcal{G}(1,n) \subset \mathbb{P}^N = \mathbb{P}(\Lambda^2(V))$.

22.3 Blowups

Setup.

$$Y \subset X.$$

$Bl_Y(X) = \tilde{X}$ (blowup of Y with respect to X).

$\pi : \tilde{X} \to X$ is a birational regular map such that $\pi_{|\tilde{X}-\pi^{-1}(Y)} \to X - Y$ is an isomorphism.

Fibers of π over Y will be positive dimensional (In fact will be projective spaces).

General Construction:

$X \subset \mathbb{A}^n$, $Y \subset X$, $\mathfrak{I}(Y) = (f_0, \dots, f_r)$, $f_i \in k[X_1, \dots, X_n]$. We have a rational map

$$\phi : \mathbb{A}^n \to \mathbb{P}^r$$

given by

$$\phi(x) = [f_0(x), \ldots, f_r(x)].$$

If $U = X - Y$ then ϕ is regular on U.

Set

$$\Gamma = \{(p, \phi(p)) : p \in U\} \subset U \times \mathbb{P}^r \subset X \times \mathbb{P}^r.$$

the graph of ϕ.

Let π_1 and π_2 be the projections on $X \times \mathbb{P}^r$ onto the first and second factors respectively. $\pi_1 = \pi : \Gamma \to U$ is an isomorphism.

Next, we'll define $\tilde{X}$ to be the closure of $\Gamma \subset X \times \mathbb{P}^r$.

$$\begin{array}{ccccc} \Gamma = \pi^{-1}(U) & \longrightarrow & \tilde{X} \subset X \times \mathbb{P}^r & & \\ \downarrow & & \downarrow{\scriptstyle \pi_1 = \pi} & \searrow^{\pi_2} & \\ U & \longrightarrow & X & & \mathbb{P}^r \end{array}$$

Thus, we have constructed the required map $\pi : \tilde{X} \to X$ such that $\pi_{|\pi^{-1}(U)} : \pi^{-1}(U) \to U$ is an isomorphism.

$\tilde{X} = Bl_Y(X)$ the blowup of X along the locus $(f_0, \ldots, f_r)$.

Remarks:

$\tilde{X}$ is birational to X, in particular, it has the same dimension.

Example 53. *$X = \mathbb{A}^2$, $Y = 0$. $f_0 = x, f_1 = y$.*

$$\phi : \mathbb{A}^2 \to \mathbb{P}^1$$

$$(x, y) \longmapsto [f_0(x, y) : f_1(x, y)] = [x : y].$$

So ϕ is really defined on $\mathbb{A}^2 - \{0\} = U$.

$$\Gamma = \{\big((x, y), [x : y]\big) : (x, y) \neq (0, 0) \subset U \times \mathbb{P}^1\}$$
$$= \{\big((x, y), [u : v]\big) : xv - yu = 0, (x, y) \neq (0, 0) \subset \mathbb{A}^2 \times \mathbb{P}^1\}.$$

Take the closure of Γ in $\mathbb{A}^2 \times \mathbb{P}^1$. Then

$$\begin{array}{c} \tilde{X} = \{\big((x, y), [u : v]\big) : xv - yu = 0\} \subset \mathbb{A}^2 \times \mathbb{P}^1 \\ \downarrow{\scriptstyle \pi} \\ X \end{array}$$

$$\tilde{X} = \bar{\Gamma} \subset \{\big((x, y), [u : v] : xv - yu = 0\}$$
$$= \Gamma \cup \{(0, 0) \times \mathbb{P}^1\}.$$

Does it happen that every $\big((0, 0), [u : v]\big) \in \bar{\Gamma}$?
Define $\chi : \mathbb{A}^1 \to \mathbb{A}^2 \times \mathbb{P}^1$ by

$$t \mapsto (tu, tv, [u : v])$$

$$\chi(\mathbb{A}^1 - \{0\}) \subset \Gamma$$

$$\chi(0) = \big((0,0) \times [u:v]\big)$$

So

$$\begin{array}{ccl} \tilde{X} & & = Bl_{(0,0)}(\mathbb{A}^2) = \{(x,y) \times [u:v] : xv = yu\} \subset \mathbb{A}^2 \times \mathbb{P}^1 \\ \downarrow \pi & & \\ \mathbb{A}^2 & & \end{array}$$

$$\pi^{-1}\big((x,y)\big) = (x,y) \times [x:y]$$
$$\pi^{-1}(0) = \mathbb{P}^1$$

We can think of the blowup of $\mathbb{A}^2$ *at the origin as* $\mathbb{A}^2 - \{(0,0)\} \cup \mathbb{P}^1$ *with the* $\mathbb{P}^1$ *located at the origin.*

Remark 22.2. *As a final remark, consider the following situation:*

$$\begin{array}{ccc} U & \subset & X \\ & \searrow^{\phi} & \\ & \mathbb{P}^r & \end{array}$$

$$\phi = [f_0 : \cdots : f_r]$$

ϕ *cannot, in general, be extended to* X*. But we have the following situation:*

$$\begin{array}{ccccc} X & \subset & \tilde{X} & \subset & X \times \mathbb{P}^r \\ & \swarrow & & \searrow & \\ & X & & \mathbb{P}^r & \end{array}$$

by construction ϕ *can be extended from* $\tilde{X}$ *to* $\mathbb{P}^r$*. Call it* $\tilde{\phi}$ *and* $\tilde{\phi_{|U}} = \phi$*. (Universal procedure for extending rational maps).*

23 4/4

In this lecture we develop a correspondence between the categories of smooth curves, C, (over an algebraically closed field, k throughout) and finitely generated field extensions, K/k, of transcendence degree 1. Of course, we have not yet defined the term, *curve*, so we should start there.

Definition 37. *A curve is a smooth algebraic variety of dimension* 1.

We have already worked on the forward direction of our correspondence: given a curve C/k, we assign to it the field $K = k(C)$, its function field. Note the assumption that C be a curve includes the hypothesis that this field to have transcendence degree 1. Under this correspondence, a point on the curve is sent to the subring $O_{C,p}$, the local ring at that point, which is a DVR with valuation $ord_p : K^* \to \mathbb{Z}$. Moreover, $O_{C,p} = \{f \in K : ord_p(f) \geq 0\} \cup \{0\}$, the valuation ring for ord_p.

Recall that a Dedekind domain is an integrally closed Noetherian domain of dimension 1, meaning that all non-zero prime ideals are maximal.

Theorem 23.1. *Let R be a domain of dimension 1. Then R is Dedkind if and only if, for all $p \in MSpec(R)$, R_p is a DVR.*

Proof. Recall that a local ring R_p is a DVR if and only if it is regular, which is equivalent to its maximal ideal being principal, which is also equivalent to R_p being integrally closed. Thus the theorem reduces to: R is integrally closed if and only if R_p is integrally closed for all $p \in MSpec(R)$. (This is a familiar sort of local to global relationship that holds for several of our favorite properties in commutative algebra. For example, a ring is Noetherian if and only if all of its localizations are Noetherian.) The forward direction of this assertion is a simple check and will be omitted. The reverse direction can be broken down into several easy steps, some of which have been on the homework. Note that

$$R = \bigcap_{p \in MSpec(R)} R_p \subset Q(R).$$

Also, note that the intersection of any collection of integrally closed localization of R is integrally closed. □

Theorem 23.2. *Let R be a Dedekind domain, $Q(R) = K$, and L/K be a finite field extension. Then if B is the integral closure of R in L, B is also Dedekind. Moreover, if R is a finitely generated k-algebra, then B is a finitely generated k-algebra as well.*

The proof is difficult and omitted. This theorem does, however, motivate the definition of a Dedekind domain, which heretofore seemed arbitrary. The theorem tells us that this is a stable property under a crucial procedure of commutative algebra. Also, since $\mathbb{Z}$ is a Dedekind domain, the theorem tells us that its integral closure in any finite extension of $\mathbb{Q}$ is Dedekind as well, a fact of great interest to number theorists. We generally do not consider $\mathbb{Q}$ as a possible field in this course but mention it in passing. Now we recall a theorem from last week:

Theorem 23.3. *Suppose K/k is a finitely generated field extension of transcendence degree 1 and $y \in K$. Then there exist only finitely many valuations of K/k, v, such that $v(y) < 0$.*

We will not reprove this, but we will rekindle our intuition for it. Let $K = k(\mathbb{P}^1) = k(t)$ and $y = \frac{f(t)}{g(t)} \in K$. The valuations for which $v(y) < 0$ correspond to the zeros of g, of which there are certainly only finitely many.

Theorem 23.4 (Fundamental Construction). *Let v be a discrete valuation of K/k. Then there exists a smooth curve C_v and a point $p \in C_v$ such that $K = k(C_v)$ and $v = ord_p$.*

Proof. Denote by (R_v, m_v) the valuation ring for v with maximal ideal m_v. Take an element $y \in m_v$. Consider $k[y] \subseteq K$. This is, as the notation suggests, a polynomial ring because the element y cannot satisfy any polynomials over the algebraically closed field k. Then $k[y] \subseteq k(y) \subseteq K$, and $k[y]$ is finitely genreated over k (as an algebra) and Dedekind being polynomials in a single variable over an algebraically closed field. Because K/k is of transcendence degree 1, $[K : k(y)] < \infty$. Let B_y be the integral closure of $k[y]$ in K. By Theorem 23.3, B_y is then a Dedekind domain and a finitely generated k-algebra. Any such ring is the affine coordinate ring of a variety, C_v. Any localization at a maximal ideal of B_y then produces a DVR, which is regular, hence C_v is smooth. It is dimension 1, so C_v is a curve.

Note that $k[y] \subseteq R_v$, and the latter is integrally closed, so $B_y \subseteq R_v$. Let $n = m_v \cap B_y$, which is then a prime ideal of B_y. Note that $y \in n$, so $n \neq (0)$, and thus n is maximal. Also note that B_y and R_v have the same field of fractions, though this is non-trivial. Then $(R_v, m_v) = ((B_y)_n, n(B_y)_n) = (O_{C_v,p}, m_p)$, and we have found p. □

Note that R, or really v, did not play a crucial role in this construction until the very end, and we claim to have actually constructed a curve such that all but finitely many of the valuations on K, the ones with $v(y) \geq 0$, correspond to the order of vanishing of some point $q \in C$.

Scholium 23.5. *Let C_v be as in the previous theorem. Then all valuations w of K/k such that $w(y) \geq 0$ are realized as ord_q for some point $q \in C_v$.*

Proof. Let w be a valuation on K/k such that $w(y) \geq 0$ with corresponding valuation ring (R_w, m_w). Because $w(y) \geq 0$, we have that $y \in R_w$. If $y \in m_w$, then we can use B_y to construct a curve for w. Then $C_w = C_v$ and we are done. If $w(y) = 0$, then denote by $\tilde{c}$ the class of y in $k = R_w/m_w$ and c a lift of $\tilde{c}$. Then let $y' = y - c \in m_w$. Then $k[y] = k[y']$ as subrings of K. This reduces us to the previous case, because $B_{y'} = B_y$ again. □

Hence by Theorem 23.3 for any extension K/k we can get a finite collection of curves such that every valuation on the field corresponds to the order of vanishing at (exactly, as we'll see later) one point on (each of at least) one of the curves. We will not have time to complete the construction, but we will proceed from this point to glue these curves together into one projective curve where there is a bijection between valuations and points.

Recall:

Lemma 23.6 (Algebraic L'Hopital). *Suppose C is a curve and $\phi : C \dashrightarrow Y$ is a rational map to a projective variety. Then there exists a unique extension $\tilde{\phi} : C \to Y$.*

Using this result, we can prove that:

Theorem 23.7. *Every smooth curve is quasi-projective, namely open in a projective curve.*

Proof. Note that C can be covered by affines, $C_i \subset \mathbb{A}^{n_i} \subset \mathbb{P}^{n_i}$. Let Y_i be the closure of C_i in $\mathbb{P}^{n_i}$ and $j_i : C_i \hookrightarrow Y_i$ be the embeddings. Then the j_i are rational maps on C into projective spaces, so by Theorem 23.6 there exist unique extensions of the j_i to $f_i : C \to Y_i$. The f_i are not in general embeddings any longer. However, we can let

$$f = (f_i) : C \to \prod Y_i \subset \prod \mathbb{P}^{n_i} \overset{Segre}{\hookrightarrow} \mathbb{P}^N.$$

Let $Y = \overline{f(C)}$. For simplicity, we will assume $C = C_1 \cup C_2$. Then $f|_{C_1}(y) = (y, f_2(y))$ and $f|_{C_2}(y) = (f_1(y), y)$. Then, locally, f^{-1} is just the projection onto one of its coordinates, which is a local isomorphism. Suppose $f(x) = f(y) = z$. Then f^* sends $O_{Y,z}$ to $O_{C,x}$ and $O_{C,y}$ as subrings of $k(C)$. But such a subring determines the point in C. Hence $x = y$, so f is injective. Thus f is an isomorphism of C with its image, which is open in its closure in $\mathbb{P}^N$, making C quasi-projective. There are many more details on this in the next lecture. □

Corollary 23.8. *Any smooth curve is birational to a projective curve.*

Proof. By the previous theorem, we can assume C is embedded in $\mathbb{P}^N$. Then its closure, $\overline{C}$ is a projective curve, and $\overline{C} \setminus C$ is a finite collection of points, so the inclusion is the birational isomorphism. $\square$

Again, our next goal will be to glue the curves with points corresponding to valuations on a field in such a way that there is a bijection between the valuations and points. apr6

24 Noramalization continued - 4/8 - Chakhad

Let X be a variety. There exists a variety $\widetilde{X}$ and a birational map $\nu : \widetilde{X} \to X$, that we call normalization, with the following properties:

1) The restriction of ν to $\nu^{-1}(X_{\text{reg}})$ is an isomorphism, where X_{reg} is the set of smooth points of X.

2) $\dim X = 1 \implies \widetilde{X}$ is smooth, since for a local ring of dimension 1, being DVR is equivalent to being regular, which is in turn equivalent to being integrally closed.

3) If $U \subseteq X$ is an open affine, then so is $\widetilde{U} := \nu^{-1}(U)$, and $\mathcal{O}_{\widetilde{X}}(\widetilde{U})$ is the integral closure of $\mathcal{O}_X(U)$ in $k(X)$.

Example 54. *(Node)*
Consider $X = \{(x, y) \in \mathbb{A}^2 : y^2 = x^3 + x^2\}$. The coordinate ring of X is $R := A(X) = k[x, y]/(y^2 - x^3 - x^2)$. Let $\widetilde{R} \subseteq Q(R)$ be the integral closure of R. Dividing both sides of the defining polynomial equation of X by x^2, and letting $t := y/x$, we obtain $t^2 = x + 1$; hence $t \in \widetilde{R}$. We claim that $\widetilde{R} = k[x, y, t]/(t^2 - x - 1, y^2 - x^3 - x^2, tx - y)$. The corresponding variety $\widetilde{X}$ (i.e. the common zero locus in $\mathbb{A}^3$ of the polynomials $x - t^2 + 1$ and $y - t^3 + t^2$) is clearly a smooth curve. The normalization map $\nu : \widetilde{X} \to X$ is given by $\nu(x, y, t) = (x, y)$, hence we have $\nu^{-1}(0) = \{(0, 0, 1), (0, 0, -1)\}$. In this example we see that the normalization of the curve X is its strict transform in the blowup of $\mathbb{A}^2$ at the origin.

Example 55. *(Cusp)*
Now, consider $Y = \{(x, y) \in \mathbb{A}^2 : y^2 = x^3\}$. The coordinate ring of Y is $R = k[x, y]/(y^2 - x^3)$. Adjoining the new variable $t := y/x$, we obtain $\widetilde{R} = k[x, y, t]/(t^2 - x, y^2 - x^3, y - xt)$. Hence the normalization of the cuspidal curve is the zero locus in $\mathbb{A}^3$ of the polynomials $x - t^2$ and $y - t^3$, which is the twisted cubic curve in $\mathbb{A}^3$.

Line Bundles and Invertible Sheaves:

Definition 38. *Let X be a variety. A* line bundle *L on X consists of the following data:*
a) an open cover $\{U_i\}$ of X,
b) and a set of regular functions $\{g_{ij} \in \mathcal{O}_X^*(U_i \cap U_j)\}$,
satisfying the cocycle condition*:*

$$\forall\, i, j, k \qquad g_{ik} = g_{ij} g_{jk} \quad \text{on} \quad U_{ijk} := U_i \cap U_j \cap U_k. \tag{1}$$

Remark 24.1. *The cocycle condition says in particular that $g_{ii} = 1_{\mathcal{O}_X(U_i)}$, and that $g_{ji} = g_{ij}^{-1}$, for all i and j.*

Remark 24.2. *A line bundle on X is equivalently defined as a triple $E \to^{\pi} X$, where E is a variety, and π is a surjective morphism, such that E looks locally like $U \times \mathbb{A}^1$ for some open set $U \subseteq X$. More precisely, there exists an open covering $\{U_i\}$ of X and isomorphisms $\phi_i : U_i \times \mathbb{A}^1 \to \pi^{-1}(U_i)$, called* local trivializations*, such that the following diagrams are commutative and such that for all $x \in U_i$, $\phi_i|_{\{x\}\times\mathbb{A}^1} : \mathbb{A}^1 \to E(x)$ is a linear isomorphism, where $E(x) := \pi^{-1}(x)$.*

Let $g_{ij} : U_{ij} \to \mathrm{GL}(\mathbb{A}^1)$ be the map given by $g_{ij}(x)(\lambda) = \phi_j^{-1} \circ \phi_i(x, \lambda)$, where $U_{ij} := U_i \cap U_j$, and $(x, \lambda) \in U_{ij} \times \mathbb{A}^1$. Now, $\mathrm{GL}(\mathbb{A}^1) = k^*$, so that $g_{ij} : U_{ij} \to k^*$. The g_{ij} are called transition functions*, and for each i and j, we have a commutative triangle diagram where the dashed arrow is given by $(x, \lambda) \mapsto (x, g_{ij}(x)\lambda)$. From the definition of the g_{ij}, it is easy to check that they satisfy the cocycle condition.*

Sections of a Line Bundle:

Definition 39. *Let X be a variety and let $L \to X$ be a line bundle given by $\{U_i, g_{ij}\}$. A* section *of $L \to X$ over an open set $U \subseteq X$ is a collection of regular function $\{s_i \in \mathcal{O}_X(U \cap U_i)\}$ such that $s_j = g_{ij}s_i$ on $U \cap U_{ij}$.*

Remark 24.3. *In topology, a section of $E \to X$ over $U \subseteq X$ is a map $s : U \to E$ such that $\pi \circ s = \mathrm{id}_U$. To show that a section in the sense of the definition gives rise to a section in the usual topological sense, let $\{s_i \in \mathcal{O}_X(U \cap U_i)\}$ be a section. Define $s : U \to L$ by $s|_{U_i}(x) = \phi_i(x, s_i(x))$. That s is well defined can be seen from the following equalities*

$$\begin{aligned} \phi_j(x, s_j(x)) &= \phi_j(x, g_{ij}(x)s_i(x)) \\ &= \phi_i(x, s_i(x)) \end{aligned}$$

where the first equality is by definition of a section, and the second equality follows from the commutativity of the second triangle diagram of the previous remark. That s is a section in the sense given at the beginning of this remark follows from the commutativity of the first triangle diagram of the previous remark.

If $L \to X$ is a line bundle, we define the *sheaf of sections* $\mathcal{L}$ of $L \to X$ as follows. For U open in X, let $\mathcal{L}(U) = \{\text{sections of } L \to X \text{ over } U\}$; the restriction maps are the usual restrictions of functions. It is easy to check that $\mathcal{L}$ is actually a sheaf. It turns out that for U open in X, $\mathcal{L}(U)$ is an $\mathcal{O}_X(U)$-module with scalar multiplication given by

$$fs = \{fs_i \in \mathcal{O}_X(U \cap U_i)\}$$

for $s = \{s_i \in \mathcal{O}_X(U \cap U_i)\} \in \mathcal{L}(U)$ and $f \in \mathcal{O}_X(U)$. Moreover, if V is an open subset of U, we have

$$fs|_V = f|_V s|_V$$

for all $s = \{s_i \in \mathcal{O}_X(U \cap U_i)\} \in \mathcal{L}(U)$ and $f \in \mathcal{O}_X(U)$. Hence we say that $\mathcal{L}$ is an $\mathcal{O}_X$*-module*.

Definition 40. *Let $\mathcal{L}$ be an $\mathcal{O}_X$-module. We say that $\mathcal{L}$ is an* invertible sheaf *if there exists an open cover $\{U_i\}$ of X together with isomorphisms $\phi_i : \mathcal{L}|_{U_i} \to \mathcal{O}_X|_{U_i}$ of $\mathcal{O}_{U_i}$-modules, and functions $f_{ij} \in O_X^*(U_{ij})$ such that $\phi_i(s) = f_{ij}\phi_j(s)$ for all $s \in \mathcal{L}(U \subseteq U_{ij})$. (Note that this definition implies the cocycle condition $f_{ij}f_{jk} = f_{ik}$ on U_{ijk}.)*

Thus, to a line bundle $L \to X$ we have associated an $\mathcal{O}_X$-module $\mathcal{L}$, which is an invertible sheaf.

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26 The Picard Group (cont.)

Recall from last time that if we have line bundles $\mathcal{L}$ and $\mathcal{M}$ on a variety X, we have a natural tensor product $\mathcal{L} \otimes \mathcal{M}$,

$$\mathcal{L} = \{(U_i, g_{ij})\} \qquad \mathcal{M} = \{(U_i, h_{ij})\}$$
$$\mathcal{L} \otimes \mathcal{M} = \{(U_i, g_{ij}h_{ij})\}.$$

As we noted earlier, we can refine our covers of $\mathcal{L}$ and $\mathcal{M}$ until they are the same, so we can make the assumption that we already have $\mathcal{L}$ and $\mathcal{M}$ defined over the same covering U_i of X. Why do we use a *tensor* product? Note that $\mathcal{L}$ and $\mathcal{M}$ are invertible $\mathcal{O}_X$-modules, so in particular we have

$$\mathcal{L}|_U = f\mathcal{O}_U \qquad f \in k(X)$$
$$\mathcal{M}|_U = g\mathcal{O}_U \qquad g \in k(X)$$
$$\mathcal{L} \otimes \mathcal{M}|_U = fg\mathcal{O}_U = f\mathcal{O}_U \otimes_{\mathcal{O}_U} g\mathcal{O}_U$$

then $\mathcal{L} \otimes \mathcal{M}$ really is the tensor product as $\mathcal{O}_X$-modules.

26.1 Pullbacks of line bundles

Suppose we have a morphism $f : X \to Y$, and $\mathcal{L} = \{(U_i, g_{ij})\}$ is a line bundle on Y. We define $f^*\mathcal{L}$ to be the line bundle on X given by

$$f^*\mathcal{L} = \{(f^{-1}(U_i), f^*(g_{ij}))\}$$

where

$$f^* : \mathcal{O}_Y(U_{ij}) \to \mathcal{O}_X(f^{-1}(U_{ij}))$$

is the induced map. Note that the cocycle condition is satisfied because f^* is a ring homomorphism.

Remark 26.1. *f^* pulls back global sections of $\mathcal{L}$ to $f^*\mathcal{L}$:*

$$f^* : \Gamma(Y, \mathcal{L}) \to \Gamma(X, f^*\mathcal{L})$$
$$s \mapsto f^*s$$

To see this, note that if $s = \{s_i \in \mathcal{O}_Y(U_i)\}$, then $f^*s = \{f^*s_i \in \mathcal{O}_X(f^{-1}(U_i))\}$ and again the compatibility conditions are still satisfied.

Remark 26.2. *Note that f^* is a group homomorphism of the Picard groups, since*

$$f^*(\mathcal{L} \otimes \mathcal{M}) = f^*\mathcal{L} \otimes f^*\mathcal{M}.$$

Last time on $\mathbb{P}^n$ we defined $\mathcal{O}_{\mathbb{P}^n}(r)$ line bundles using the standard cover of $\mathbb{P}^n$ given by $U_i = \{x_i \neq 0\}$ by

$$g_{ij} = \left(\frac{x_i}{x_j}\right)^r$$

and we noted that

$$\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(r)) = \{\text{homogenous polynomials of degree } r \text{ in } x_0, \dots, x_n\}$$

so we have

$$\operatorname{Pic}(\mathbb{P}^n) = \mathbb{Z}[\mathcal{O}_{\mathbb{P}^n}(1)]. \tag{2}$$

There is no torsion here.

Let's do something funny. Let Y be the complement of an irreducible curve in $\mathbb{P}^2$, and $i : Y \hookrightarrow \mathbb{P}^2$ its inclusion. Then we have a pullback homomorphism

$$i^* : \operatorname{Pic}(\mathbb{P}^2) \to \operatorname{Pic}(Y).$$

Suppose Y is the complement of an irreducible conic. Let's figure out $\operatorname{Pic}(Y)$. So $Y = \{[x, y, z] : f_2(x, y, z) \neq 0\}$ where $f_2 \in k[x, y, z]$ is a homogenous polynomial of degree 2. We have the line bundle $\mathcal{O}_Y(2) = i^*\mathcal{O}_{\mathbb{P}^2}(2)$. Note that the transition functions remain the same. We claim that $\mathcal{O}_Y(2) = \mathcal{O}_Y$ (i.e., $\mathcal{O}_Y(2)$ is trivial) but $\mathcal{O}_Y(1) \neq \mathcal{O}_Y$. This implies that $\mathcal{O}_Y(1)$ is a torsion element of order 2 in $\operatorname{Pic}(Y)$. To see these claims, recall that a bundle is trivial if we can produce an everywhere nonzero section. Note that $s = f_2 \in \Gamma(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$, so $s|_Y \in \Gamma(Y, \mathcal{O}_Y(2))$ and $s(y) \neq 0$ for all $y \in Y$, so $\mathcal{O}_Y(2)$ is indeed trivial. To show that $\mathcal{O}_Y(1) \neq \mathcal{O}_Y$ we need to show that any section must vanish somewhere on Y. First, we show that $i^* : \Gamma(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)) \xrightarrow{\sim} \Gamma(Y, \mathcal{O}_Y(1))$ is an isomorphism. Then we note that any $s \in \Gamma(\mathcal{O}_{\mathbb{P}^2}(1))$ vanishes along a line in $\mathbb{P}^2$ (linear polynomials), but since Y^c is nondegenerate, no line in $\mathbb{P}^2$ is contained in Y^c, so the zero locus of any section will have to meet Y.

27 Maps to Projective Space

Suppose X is a variety, $L \to X$ a line bundle and $s^0, \dots, s^n \in \Gamma(X, L)$ are sections which do not vanish simultaneously anywhere. Let

$$\phi : X \to \mathbb{P}^n$$
$$\phi = [s^0, \dots, s^n].$$

We do we mean by this? Let $L = \{(U_i, g_{ij})\}$. Then $s^\alpha = \{s_i^\alpha \in \mathcal{O}_X(U_i)\}$ for $0 \le \alpha \le n$. Then

$$\phi|_{U_i}(x) = [s_i^0(x), \dots, s_i^\alpha(x), \dots, s_i^n(x)]$$

and $s_j^\alpha = g_{ij} s_i^\alpha$. We need to check that $\phi|_{U_{ij}}$ makes sense:

$$[s_j^0(x), \dots, s_j^n(x)] = [g_{ij}(x)s_i^0(x), \dots, g_{ij}(x)s_i^n(x)] = [s_i^0(x), \dots, s_i^n(x)]$$

since this is just a rescaling, so the definitions agree on the overlap, and $\phi : X \to \mathbb{P}^n$ is well-defined everywhere, and it is an algebraic map because $\phi|_{U_i}$ is given by regular functions, so it is morphism.

Theorem 27.1. *$\phi^*(\mathcal{O}_{\mathbb{P}^n}(1)) \simeq L$.*

Recall that $\phi^* : \Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) \to \Gamma(X, L)$, where $\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ is the set of linear functions in $x^0, \dots, x^n$. We claiming that $\phi^*(x^\alpha) = s^\alpha$. So given *any* morphism, pulling back to X on $\mathcal{O}_{\mathbb{P}^n}(1)$ gives us the sections which "define" the map.

Proof. Let $V^\alpha = \{x^\alpha \neq 0\}$ be our standard cover of $\mathbb{P}^n$ with open affines, and write $\mathcal{O}_{\mathbb{P}^n}(1) = \{(V^\alpha, g_{\alpha\beta} = x^\alpha/x^\beta)\}$. Then

$$\phi^*\mathcal{O}_{\mathbb{P}^n}(1) = \{U^\alpha = \phi^{-1}(V^\alpha) = \{s^\alpha \neq 0\}, \phi^*(x^\alpha/x^\beta)\}.$$

We claim that this is isomorphic to our initial bundle. Note that $\phi^*(x^\alpha/x^\beta) = s^\alpha/s^\beta \in \mathcal{O}_X^\times(U^\alpha \cap U^\beta)$. For $x \in U_i \cap U^\alpha \cap U^\beta$,

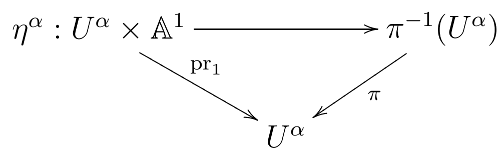
$$\frac{s^\alpha}{s^\beta}(x) = \frac{s_i^\alpha(x)}{s_i^\beta(x)}$$

and this is consistent on overlaps $U_{ij} \cap U^\alpha \cap U^\beta$:

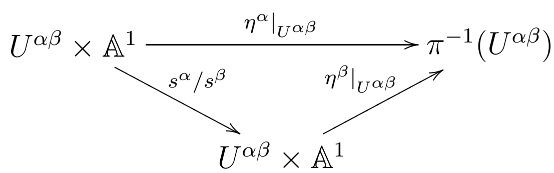
$$\frac{s_j^\alpha(x)}{s_j^\beta(x)} = \frac{g_{ij}(x)s_i^\alpha(x)}{g_{ij}(x)s_i^\beta(x)} = \frac{s_i^\alpha(x)}{s_i^\beta(x)}$$

so it is well-defined.

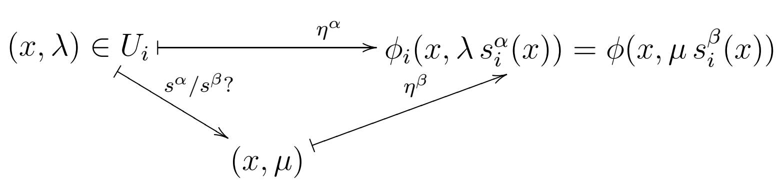
We now want to construct new trivializations:



such that the transition functions



commute. Let $\eta^\alpha(x, \lambda) = \lambda \underbrace{s^\alpha(x)}_{\neq 0}$ and it is a point in the fiber. So if $x \in U_i$, then $\eta^\alpha(x, \lambda) = \phi_i(x, \lambda\, s_i^\alpha(x))$, where ϕ_i is our original trivialization of L. We still need to check that η^α is well-defined on U_{ij}, but this is immediate:



Then

$$\mu = \lambda \frac{s_i^\alpha(x)}{s_i^\beta(x)} = \lambda \left(\frac{s^\alpha}{s^\beta}\right)(x)$$

so s^α/s^β really works (and we can remove the "?"). We claim victory:

$$\phi^*\mathcal{O}_{\mathbb{P}^n}(1) \simeq L.$$

□

Proposition 27.2. *Given any map $\phi : X \to \mathbb{P}^n$, we can create a line bundle $L \to X$ by $L = \phi^*\mathcal{O}_{\mathbb{P}^n}(1)$ with sections $s^\alpha = \phi^*(x^\alpha)$, $0 \leq \alpha \leq n$, these do not vanish simultaneously anywhere on X, and such that $\phi = [s^0, \dots, s^n]$.*

So we have an equivalence:

$$\{\phi : X \to \mathbb{P}^n\} \overset{1:1}{\longleftrightarrow} \left\{\begin{matrix}\text{Line bundles } L \to X, \text{ sections } s_0, \dots, s_n \in \Gamma(X, L) \\ \text{not vanishing simultaneously}\end{matrix}\right\}$$

Our varieties are abstract, but it's good to embed them in $\mathbb{P}^n$ every now and then (basically all the time).

28 The Picard group of a Curve

Let C be a smooth curve. Define $\mathrm{Div}(C)$ to equal the free abelian group generated by the points of C:

$$\mathrm{Div}(C) = \{n_p \cdot p \mid n_p \in \mathbb{Z}\},$$

so we're only looking at finite sums.

Definition 41. *$D = n_{p_1}p_1 + \cdots + n_{p_s}p_s$ is called a divisor of C. If $n_p \geq 0$ $\forall p \in C$, then D is an effective divisor. If $f \in k(C)^\times$ $\forall p \in C$ then $\mathcal{O}_{C,p}$ is a DVR with valuation ord_p: $k(C)^\times \to \mathbb{Z}$. Then we associate to f*

$$f \mapsto \ div(f) = (f) = \sum_{p \in C} \ ord_p(f) \cdot p.$$

Here, (f) is called the principal divisor.

How do we know that this is a finite sum?

Remark 28.1. *$\forall f \neq 0$ there are only finitely many valuations on $k(C)/k$ such that $v(y) \neq 0$. Then*

$$Div^0(C) = \{div(f) \mid f \in k^\times(0)\}$$

is a subgroup of $Div(C)$. It follow from $div(f)+div(g) = div(f \cdot g)$ that

$$Cl(C) = \frac{Div(C)}{Div^0(C)},$$

where $Cl(C)$ denotes the class group of C. Thus, this turns out to be a covering isomorphism to $Pic(C)$.

Example 56. *Consider $C = \mathbb{P}^1$ and let $f(t) = \dfrac{1}{t} \in k(C)$. Then $div(f) = \infty - 0$. Also, $div(t) = 0 - \infty$. These ∞'s and 0's are all coming from the formula $\{$ number of poles$\}$ $-$ $\{$number of zeros$\}$. Thus,*

$$div\left(\frac{t^2 - 2t}{t_3}\right) = (2) + (0) - (3) - (\infty).$$

We then see that $Div^0(\mathbb{P}^1) = \{\sum n_p \cdot p \mid \sum n_p = 0\}$.

Example 57. *If* $f = \frac{(t-a_1)\cdots(t-a_m)}{(t-b_1)\cdots(t-b_n)}$ *then*

$$div(f) = a_1 + \cdots + a_m - (b_1 + \cdots + b_n) + (n-m)\infty.$$

Example 58. *If* $D = 2\cdot(1) + 3\cdot(2) - 4\cdot(7) - \cdot(9)$, *then*

$$f = \frac{(t-1)^2(t-2)^3}{(t-7)^4(t-9)} \in k(\mathbb{P}^1).$$

We also have $Cl(\mathbb{P}^1) = \frac{\mathrm{Div}(\mathbb{P}^1)}{\mathrm{Div}^0(\mathbb{P}^1)} \cong \mathbb{Z}$ by the degree map:

$$\deg: \mathrm{Div}(\mathbb{P}^1) \to \mathbb{Z}$$

where $\deg(D) = \sum n_p$.

29 4/20 - Deblois

Notation Let C be a smooth curve.

- For $D, D' \in \mathrm{Div}(C)$, we say D and D' are *linearly equivalent*, written $D \equiv D'$ or $D \sim D'$, iff $D = D' + div(f)$ for some $f \in k(C)^*$.
- For $D = n_1p_1 + \ldots n_sp_s$, the *support* of D is $\mathrm{supp}(D) = \{p_1, \ldots, p_s\}$.

Theorem 29.1. *For* C *a smooth curve, there is a canonical isomorphism*

$$\mathrm{Cl}(C) \to \mathrm{Pic}(C)$$

Proof. We define a map $\phi:\ \mathrm{Div}(C) \to \mathrm{Pic}(C)$ with $\mathrm{Div}^0(C) \subseteq \ker(\phi)$, thus inducing a map $\phi: \mathrm{Cl}(C) \to \mathrm{Pic}(C)$.

For $D = \sum n_p \cdot p \in \mathrm{Div}(C)$, define a sheaf $\mathcal{O}_C(D)$ as follows: for $U \subset C$ open,

$$\Gamma(U, \mathcal{O}_C(D)) = \{f \in k(C) : \forall\ p \in U,\ ord_p(f) + n_p \geq 0\} \subseteq k(C)$$

This is clearly a sheaf, in fact an $\mathcal{O}_C$-module, since restrictions are inclusions inside $k(C)$. We show below that this is an invertible sheaf, hence the sheaf of sections of a line bundle.

Example Suppose $D = 0$. Then

$$\begin{aligned} \Gamma(U, \mathcal{O}_C(D)) &= \{f \in k(C)\ :\ ord_p(f) \geq 0\ \forall\ p \in U\} \\ &= \bigcap_{p \in U} \mathcal{O}_{C,p} \\ &= \mathcal{O}_C(U) \end{aligned}$$

Thus $\mathcal{O}_C(0) = \mathcal{O}_C$.

To show $\mathcal{O}_C(D)$ is an invertible sheaf, we show it is locally isomorphic to $\mathcal{O}_C$. Let $U_0 = C - \text{supp}(D)$. For $U \subset U_0$ open, by definition

$$\Gamma(U, \mathcal{O}_C(D)) = \{f \in k(C) \ : \ ord_p(f) \geq 0 \ \forall \ p \in U\} = \mathcal{O}_C(U).$$

Hence $\mathcal{O}_C(D)|_{U_0} = \mathcal{O}_{U_0}$.

Write $D = n_1 p_1 + \ldots n_s p_s$, and for each i fix $f_i \in k(C)$ such that $ord_{p_i}(f_i) = n_i$ (for example by taking $f = t^{n_i}$, where t is a uniformizing parameter for $\mathcal{O}_{C,p_i}$). Then $div(f_i) = n_i p_i + E_i$, where p_i is not in the support of E_i. Let $U_i = C - \text{supp}(E_i) - \bigcup_{j \neq i}\{p_j\}$. Now for $U \subset U_i$ open,

$$\begin{aligned} \Gamma(U, \mathcal{O}_C(D)) &= \{f \in k(C) \ : \ ord_{p_i}(f) + n_i \geq 0 \text{ and } ord_p(f) \geq 0 \ \forall \ p \neq p_i\} \\ &= \{f \in k(C) \ : \ ord_{p_i}(f) + ord_{p_i}(f_i) \geq 0 \text{ and } ord_p(f) \geq 0 \ \forall \ p \neq p_i\} \\ &= \{f \in k(C) \ : \ ord_p(f f_i) \geq 0 \ \forall \ p \in U\} \\ &= \{f \in k(C) \ : \ f f_i \in \mathcal{O}_C(U)\} \\ &= \frac{1}{f_i} \cdot \mathcal{O}_C(U) \subset k(C) \end{aligned}$$

Thus multiplication by f_i gives an $\mathcal{O}_C(U)$-module isomorphism between $\Gamma(U, \mathcal{O}_C(D))$ and $\mathcal{O}_C(U)$; therefore $\mathcal{O}_C(D)|_{U_i} \cong \mathcal{O}_{U_i}$ as $\mathcal{O}_{U_i}$-modules. This shows $\mathcal{O}_C(D)$ is locally trivial and hence an element of $\text{Pic}(C)$. Define $\phi(D) = \mathcal{O}_C(D) \in \text{Pic}(C)$. By above this gives a map $\text{Div(C)} \to \text{Pic}(C)$.

Now suppose $D = div(u)$ for some $u \in k(C)$. Then for any $U \subset C$ open

$$\begin{aligned} \Gamma(U, \mathcal{O}_C(D)) &= \{f \in k(C) \ : \ fu \in \mathcal{O}_C(U)\} \\ &= \frac{1}{u} \cdot \mathcal{O}_C(U) \cong \mathcal{O}_C(U) \end{aligned}$$

Thus $\mathcal{O}_C(D)$ is globally isomorphic to $\mathcal{O}_C$, so $\phi(D) = 0 \in \text{Pic}(C)$. This gives an induced map $\phi : \ \text{CL}(C) \to \text{Pic}(C)$.

We wish to define a map $\eta : \ \text{Pic}(C) \to \text{Cl}(C)$. In order to do so, consider a line bundle $L \to C$ and suppose there exists a nonzero section $s \in \Gamma(C, \mathcal{L})$,

$$s = \{s_i \in \mathcal{O}_C(U_i) \ : \ s_j = s_i g_{ij}\}.$$

Then we may define $Z(s) = \sum_{p \in C} ord_p(s) \cdot p \quad \in \text{Div}(C)$; where $ord_p(s) := ord_p(s_i)$ if $p \in U_i$.

In general we must consider *rational sections* $s : C \to L$,

$$s = \{s_i \in k(C) \ : \ s_j = s_i g_{ij}\}.$$

Let $G = \{(L, s) \ : \ L \in \text{Pic}(C), \ s \text{ a rational section of L}\}$, then we will induce $\eta : \text{Pic}(C) \to \text{Cl}(C)$ via the following diagram:

$$\begin{array}{ccc} G & \xrightarrow{Z} & \mathrm{Div}(C) \\ \downarrow & & \downarrow \\ \mathrm{Pic}(C) & \xrightarrow{\eta} & \mathrm{Cl}(C) \end{array}$$

For a rational section s, analogously define $Z(s) = \sum_{p \in C} ord_p(s) \cdot p$, where $ord_p(s) = ord_p(s_i)$ if $p \in U_i$. (Note that the map Z depends on the line bundle L, however this is suppressed in the notation.)

NB $Z(s) \neq div(s_i)$, for the g_{ij} may have zeros and poles outside U_{ij}.

Observe that the linear equivalence class of $Z(s)$ does not depend on the choice of rational section s: if t is another rational section, then $\frac{s}{t}$ is a well-defined rational function, for

$$\frac{s_j}{t_j} = \frac{s_i g_{ij}}{t_i g_{ij}} = \frac{s_i}{t_i}$$

Therefore $Z(s) = Z(t) + div(s/t)$. There is thus an induced map η given by $\eta(L) = Z(s) \in \mathrm{Cl}(C)$ for any rational section s of L.

We now show that η and ϕ are isomorphisms by showing that they are inverses of each other. For a divisor $D = n_1 \cdot p_1 + \ldots + n_k \cdot p_k$, recall that $\phi(D) = \mathcal{O}_C(D)$ was defined by

$$\Gamma(U, \mathcal{O}_C(D)) = \{f \in k(C) \ : \ ord_p(f) + n_p \geq 0 \ \forall \ p \in U\}$$

for $U \subset C$ open. Note that the rational sections of $\mathcal{O}_C(D)$ are in one-to-one correspondence with elements of $k(C)$: $f \in k(C)$ determines a section s with $s_0 = f$. Now consider the rational section s with $s_0 = 1 \in k(C)$; then we claim that $Z(s) = D$. Clearly for any point $p \in U_0$, $ord_p(s) = ord_p(1) = 0$. On U_i, $s_i = f_i \cdot 1 = f_i$, where f_i as above is a function so that $ord_{p_i}(f_i) = n_{p_i}$. Thus $ord_{p_i}(s) = ord_{p_i}(s_i) = ord_{p_i}(f_i) = n_{p_i}$, and hence $\eta \circ \phi$ is the identity.

Now consider a line bundle $L = \{U_i, g_{ij}\}$. Define a rational section s by $s_0 = 1$; then for $i > 0$, $s_i = g_{0i} s_0 = g_{0i}$. Then $\phi \circ \eta(L) = \mathcal{O}_C(Z(s))$. For $U \subset C$ open,

$$\begin{aligned} \Gamma(U, \mathcal{O}_C(Z(s))) &= \{f \in k(C) \ : \ ord_p(f) + ord_p(s) \geq 0 \ \forall \ p \in U\} \\ &= \{f \in k(C) \ : \ ord_p(f) + ord_p(g_{0i}) \geq 0 \ \forall \ p \in U \cap U_i\} \\ &= \{f \in k(C) \ : \ ord_p(f \cdot g_{0i}) \geq 0 \ \forall \ p \in U \cap U_i\} \\ &= \{f \in k(C) \ : \ f \cdot g_{0i} \in \mathcal{O}_C(U \cap U_i)\} \\ &= \Gamma(U, \mathcal{L}). \end{aligned}$$

Therefore $\phi \circ \eta$ is the identity, and $\mathrm{Cl}(C)$ is isomorphic to $\mathrm{Pic}(C)$.

□

30 4/25 - Luxton

Throughout these notes C will always be a smooth, projective curve. We have a well-defined map $deg : Pic(C) \to \mathbb{Z}$ where $deg \mathcal{O}_C(D) = deg D$.

Notation: Let $D, D' \in Div(C)$. Then $D \leq D'$ iff $D - D'$ is effective.

31 The k-vector spaces $\Gamma(C, \mathcal{O}_C(D)$

Our ultimate goal is to answer the Riemann-Roch problem, that is, to understand $\Gamma(C, \mathcal{O}_C(D))$ for all $D \in Div(C)$. Since these are all finite dimensional vector spaces over k, we're asking to find the dimension. For example, if $D = 0$, $\Gamma(C, \mathcal{O}_C(D)) = k$.

Lemma 31.1. *If $\mathcal{L}$ is a line bundle and $\Gamma(C, \mathcal{L}) \neq 0$, $deg\mathcal{L} \geq 0$. Moreover, if $deg\mathcal{L} = 0$, $\mathcal{L} \simeq \mathcal{O}_C$.*

Proof. Let $0 \neq s \in \Gamma(C, \mathcal{L})$. Then $Z(s) = \sum_p ord_p(s) \cdot p$ is an effective divisor so $degZ(s) = deg\mathcal{L} \geq 0$. If $deg\mathcal{L} = 0$, $\mathcal{L}$ has a regular, nowhere vanishing global section and so $\mathcal{L} \simeq \mathcal{O}_C$ and $s \in k$. □

Proposition 31.2. 1) *If $D \leq D'$, $\Gamma(C, \mathcal{O}_C(D)) \hookrightarrow \Gamma(C, \mathcal{O}_C(D'))$.*

2) *$dim_k\Gamma(C, \mathcal{O}_C(D)) \leq \infty$. In fact, $dim_k\Gamma(C, \mathcal{O}_C(D)) \leq deg(D) + 1$.*

Proof. 1) is obvious if we view both as living inside $k(C)$, the function field of C.

For 2), by the lemma, we may assume $degD \geq 0$. In this case, $\Gamma(C, \mathcal{O}_C(D)) \neq 0$ since we may take $1 \in k(C)$.

Now write $D = p_1 + \ldots + p_s$ where the p_i's are not necessarily distinct. We then have $0 \hookrightarrow \Gamma(\mathcal{O}_C(p_1) \hookrightarrow \ldots \hookrightarrow \Gamma(\mathcal{O}_C(p_1 + \ldots + p_r)$ so it is enough to show that for all $E \in Div(C), p \in C$, $dim_k \frac{\Gamma(\mathcal{O}_C(E+p))}{\Gamma(\mathcal{O}_C(E))} \leq 1$.

Write $E = ap + E'$ where $p \notin Supp(E')$ and $a < 0$. Fix a uniformizing parameter $t \in \mathcal{O}_{C,p}$ at p. Define $\phi : \Gamma(\mathcal{O}_C(D)) \to k$ by $s \mapsto (st^{a+1})(p) \in k$. Note that $(st^{a+1})(p) \in \mathcal{O}_{C,p} \to \mathcal{O}_{C,p}/\mathfrak{m}_p \simeq k$. But $ker\phi = \{s \in \Gamma(\mathcal{O}_C(D)) \mid ord_p s = a\} = \Gamma(\mathcal{O}_C(E)$ so $\frac{\Gamma(\mathcal{O}_C(E+P)}{\Gamma(\mathcal{O}_C(E)} \hookrightarrow k$. □

32 Sheaf of Differentials

Let X be any variety. We'll construct a sheaf $\Omega^1_{X/k}$ which will be the algebraic analogue of the cotangent bundle from differential geometry. To carry this out, for any k-algebra R we'd like an associated module $\Omega^1_{R/k}$ so that if $U \subset X$ is open and affine, $\Gamma(U, \Omega^1_{X/k}) = \Omega^1_{\mathcal{O}_X(U)/k}$.

32.1 The Module of Differentials

First some algebra: Let R be a finitely generated k-algebra. A derivation of R is a map $d : R \to M$ where M is an R-module such that:

$$d(x+y) = dx + dy \qquad \text{(additive)} \qquad (3)$$

$$d(\lambda) = 0 \qquad \text{for all } \lambda \in k \qquad (4)$$

$$d(xy) = xd(y) + yd(x) \qquad \text{(Leibniz Rule)} \qquad (5)$$

Theorem 32.1. *There exist an R-module $\Omega^1_{R/k}$ together with a derivation $d : R \to \Omega^1_{R/k}$ satisfying the following universal property: For all derivations $D : R \to M$, there is a unique R-module morphism $\phi : \Omega^1_{R/k} \to M$, such that $D = \phi \circ d$. $\Omega^1_{R/k}$ is called the module of (Kähler) differentials.*

Proof. We simply take $\Omega^1_{R/k}$ to be the free R-module generated by $\{d\alpha\}_{\alpha \in R}$ modulo the three above relations. □

32.1.1 Example

If R is finitely generated over k by $x_1, \dots, x_n$, $\Omega^1_{R/k}$ is finitely generated over k with generators $dx_1, \dots, dx_n$. If $f(x_1, \dots, x_n) \in k[x_1, \dots, x_n]$, $df(x_1, \dots, x_n) = \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i \in \Omega^1_{R/k}$.

32.1.2 Example

If $R = k[t]$, $\Omega^1_{R/k} = k[t] < dt >$. If $R = k[x_1, \dots, x_n]$, $\Omega^1_{R/k} = \bigoplus_{i=1}^n R \cdot dx_i$, the free R-module generated by $dx_1, \dots, dx_n$. A typical $\omega = \sum a_i(x_1, \dots, x_n) \cdot dx_i$.

32.1.3 Remark

$\Omega^1_{R/k}$ behaves nicely with respect to localization: if $S \subset R$ is multiplicative, consider $S^{-1}R$. Then $\Omega^1_{S^{-1}R/k} \simeq S^{-1}(\Omega^1_{R/k})$ canonically.

32.1.4 Example

If $X \subset \mathbb{A}^n$, $A(X) = \mathcal{O}_X(X)$, and $\mathfrak{m}_p \subset A(X)$ corresponds to a point $p \in X$, $\mathcal{O}_{X,p} = A(X)_{\mathfrak{m}_p}$, $\Omega^1_{\mathcal{O}_{X,p}/k} = (\Omega^1_{A(X)/k})_{\mathfrak{m}_p}$. This is important for constructing a sheaf as it gives the stalk at each point.

32.2 Sheaf of Differentials on a Curve

Let $K = k(C)$. Then $\Omega^1_{K/k}$ is a one-dimensional vector space over K with basis given by dt for any $t \in K \backslash k$: take any $t \in K \backslash k$. Then $[K : k(t)] < \infty$. We would like for any $f \in K$, $df = (rational function) \cdot dt$.

f satisfies some polynomial $a_n(t)f^n + \dots a_1(t)f + a_0(t) = 0$ with $a_i \in k[t]$. Suppose n is minimal. Differentiating we get $g(t, f)dt + h(t, f)df = 0$ for some $g, h \in k[t, f]$, where $h(t, f)$ has degree $n - 1$ in f so that $h \neq 0$. Hence $df \in K \cdot dt$.

We call $\Omega^1_{K/k}$ the space of meromorphic differentials.

32.2.1 The Divisor of a Meromorphic Differential

Take any $0 \neq \omega \in \Omega^1_{K/k}$. We will associated to ω the divisor $div(\omega) = \sum_{p \in C} ord_p(\omega)$. We then want to define $ord_p(\omega)$.

Fix $p \in C, t \in \mathcal{O}_{C,p}$ a uniformizing parameter. We've seen that dt generates $\Omega^1_{K/k}$ and so $\omega = f dt$ for some $f \in K$. We therefore define $ord_p(\omega) = ord_p(f)$.

If $\omega, \omega' \in \Omega^1_{K/k} \backslash \{0\}$, $\omega' = f\omega$ for some $f \in K$. So $div(\omega') = div(\omega) + div(f)$ iff $div(\omega') \equiv div(\omega)$. We therefore define the canonical bundle to be $K_C = \Omega^1_C = \mathcal{O}_C(div(\omega))$ for any $\omega \in \Omega^1_{K/k} \backslash \{0\}$.

Recall from last time that we have three ways of viewing line bundles. We can look at them geometrically as locally isomorphic to products of affine lines over the base, we can see them from the point of view of their transition functions, such that, $g_{ij} \in \mathcal{O}^*_X(U_{ij})$, and finally we can view them as a sheaves of sections $L \to \mathcal{L}$ where $\mathcal{L}$ is an invertible $\mathcal{O}_X$-module.

Remark: $L \simeq X \times \mathbb{A}^1 \Leftrightarrow \mathcal{L} \simeq \mathcal{O}_X$ is trivial $\Leftrightarrow g_{ij} = u_i/u_j$ with $u_i \in \mathcal{O}^*_X(U_i)$. In other words, $\mathcal{L}$ is trivial iff $\exists s \in \Gamma(X, \mathcal{L})$ which is nowhere vanishing. For example, $s \notin \mathfrak{m}_x \subseteq$

$\mathcal{O}_{X,x}, \forall x \in X$. So $s : X \to L$ and $\pi : L \to X$ for a total space L iff $s = \{s_i \in \mathcal{O}_x(U_i) : s_j = g_{ij} s_i$ where $s_i \neq 0\}$.

Now, suppose $s(x) \neq \mathcal{O}_x$ with $(\mathcal{O}_x \in \pi^{-1}(x) \simeq k)$, where

$$\begin{array}{ccc} X \times \mathbb{A}^1 & \xrightarrow{\eta} & L \\ \downarrow{\scriptstyle Pr} & & \downarrow{\scriptstyle \pi} \\ X & = & X \end{array} \tag{6}$$

commutes and $\eta(x, \lambda) = \lambda s(x) \in \pi^{-1}(x)$, then η is an isomorphism in each fiber rescaled by λ.

Operations with Line Bundles (*Tensor Product*)

Let $\mathcal{L}, \mathcal{M}$ be two invertible $\mathcal{O}_x$-modules, which implies that $\mathcal{L} \times \mathcal{M}$ is an invertible $\mathcal{O}_x$-module as well. Now set $\mathcal{L} = \{(U_i, g_{ij})\}$ and $\mathcal{M} = \{(U_i, h_{ij})\}$, then $\mathcal{L} \otimes \mathcal{M} \overset{def}{=} \{U_i, g_{ij}, h_{ij} \in \mathcal{O}_x^*(U_{ij})\}$.

Remark: What if $\mathcal{L}$ and $\mathcal{M}$ are not trivialized over the same covering of X? Take $\mathcal{L} = \{U_i, g_{i\acute{i}} \in \mathcal{O}_x^*(U_{i\acute{i}})\}$ and $\mathcal{M} = \{V_i, h_{j\acute{j}} \in \mathcal{O}_x^*(U_{j\acute{j}})\}$, so that $\mathcal{L} \otimes \mathcal{M} = \{U_i \cap V_j, U_i \cap V_j \cap (U_{\acute{i}} \cap V_{\acute{j}}) = U_{i\acute{i}} \cap V_{j\acute{j}}\}$. So $g_{i\acute{i}} h_{j\acute{j}}|_{(U_{i\acute{i}} \cap V_{j\acute{j}})} \in \mathcal{O}_x^*(U_{i\acute{i}} \cap V_{j\acute{j}})$. Now, the Picard of X is given as the isomorphism classes of invertible $\mathcal{O}_x$-modules $\mathcal{L}$ over X, so $X = \mathcal{L} \times \mathcal{M} \mapsto \mathcal{L} \otimes \mathcal{M} \in \text{Pic}(X)$. Given the trivial bundle $\mathcal{O}_x = \{U_i, 1 \in \mathcal{O}_x^*(U_i)\}$, see that $\mathcal{L} \otimes \mathcal{O}_x = \mathcal{O}_x \otimes \mathcal{L} = \mathcal{L}$ and given the dual $\mathcal{L}^v = \{U_i, \frac{1}{g_ij} \in \mathcal{O}_x^*(U_{ij})\}$, then $\mathcal{L} \otimes \mathcal{L}^v = \mathcal{L}^v \otimes \mathcal{L} = \mathcal{O}_x$.

More examples

Recall from last time the nontrivial tautological line bundle $\mathcal{O}_{\mathbb{P}^n}(-1) = \{U_i, g_{ij} = \frac{x_j}{x_i}\}$. Thus we can define the dual $\mathcal{O}_{\mathbb{P}^n}(1) \overset{def}{=} \mathcal{O}_{\mathbb{P}^n}(-1)^v = \{U_i, \frac{x_i}{x_j}\}$. Likewise, we can extend over the tensor product as $\mathcal{O}_{\mathbb{P}^n}(r) \overset{def}{=} \mathcal{O}_{\mathbb{P}^n}(1)^{\otimes r} = \{U_i, \left(\frac{x_i}{x_j}\right)^r\}$.

A section $\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(r))$ is given as a polynomial in $n+1$ variables as $\{f(x_0 \dots x_n) \in k[x_0 \dots x_n]\}$ with f homogeneous of degree r.

Further, if $r \neq s$, then $\mathcal{O}_{\mathbb{P}^n}(r) \neq \mathcal{O}_{\mathbb{P}^n}(s)$.

Now, let $X = \mathbb{P}^1$, with $\Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) = \{(s_0, s_1) \in k[\frac{1}{t}] \times k[t], s_1 = g_{01} s_0\}$. Let $g_{01} = t$ so that $s_1 = t s_0$. Since s_0 must have degree one, we can set solutions as $s_0 = a_0 + \frac{a_1}{t}$ and $s_1 = a_0 t + a_1$. This implies that $\{a_0 t + a_1 : a_0, a_1 \in k\} \simeq k^2$.

Likewise, for $\Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(r)) = \{(s_0, s_1) \in k[\frac{1}{t}] \times k[t], s_1 = t^r s_0\}$, with transition function of $\mathcal{O}_{\mathbb{P}^1}(r)$ given by t^r as above, then $s_0 = a_0 + \frac{a_1}{t} + \dots + \frac{a_r}{t^r}$, implies $\{a_0 t^r + a_1 t^{r-1} + \dots + a_r, a_i \in k\} \simeq k^{r+1}$ after homogenizing the equation with $t = \frac{x_0}{x_1}$ as homogenous polynomials of degree r in x_0 and x_1.

Remark: $\Gamma(\mathcal{O}_{\mathbb{P}^1}(-r)) = 0$ for $r > 0$.

Now consider $\Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(3)) = \{(s_0, s_1) : s_1 = t^3 s_0\}$. Then we can set $s_0 = a_0 + \frac{a_1}{t} + \frac{a_2}{t^2} + \frac{a_3}{t^3}$ and $s_1 = a_0 t^3 + a_1 t^2 + a_2 t + a_3$. With $s = (s_0, s_1) \in \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(3))$ we have the zero locus $Z(s) = \{x \in \mathbb{P}^1, s(x) = 0\}$. Having chosen a_0, a_1, a_2, and a_3 generically, we can let $Z(s) = \{\lambda_1, \lambda_2, \lambda_3\}$ where we write $Z(s) = \lambda_1 + \lambda_2 + \lambda_3$ and $(+)$ is *not* the operation of addition here.

For example, suppose that $s = (\frac{1}{t}, t^2)$ where $s_0 = \frac{1}{t}$ and $s_1 = t^2$. Then $Z(s) = 2 \cdot 0 + 1 \cdot \infty$ which is just a formal combination of three points in $\mathbb{P}^1$.

Likewise, if $s = (t, \frac{1}{t^2})$, then $Z(s) = 1 \cdot 0 + 2 \cdot \infty$. If $s = (\frac{1}{t^s}, 1)$ then $Z(s) = 3 \cdot \infty$. And so on.

So formally, $\forall s \in \Gamma(\mathcal{O}_{\mathbb{P}^1}(3))$, $Z(s)$ is a formal combination of 3 points on $\mathbb{P}^1$. Thus, the degree of the line bundle is (3).