Apr. 18, Monday (1.5 hr class)
(NoteTEXers: Paul Fili and Zach Miner)

1. The Picard Group (cont.)

Recall from last time that if we have line bundles $\mathcal{L}$ and $\mathcal{M}$ on a variety X, we have a natural tensor product $\mathcal{L} \otimes \mathcal{M}$,

$$\mathcal{L} = \{(U_i, g_{ij})\} \qquad \mathcal{M} = \{(U_i, h_{ij})\}$$
$$\mathcal{L} \otimes \mathcal{M} = \{(U_i, g_{ij}h_{ij})\}.$$

As we noted earlier, we can refine our covers of $\mathcal{L}$ and $\mathcal{M}$ until they are the same, so we can make the assumption that we already have $\mathcal{L}$ and $\mathcal{M}$ defined over the same covering U_i of X. Why do we use a *tensor* product? Note that $\mathcal{L}$ and $\mathcal{M}$ are invertible $\mathcal{O}_X$-modules, so in particular we have

$$\mathcal{L}|_U = f\mathcal{O}_U \qquad f \in k(X)$$
$$\mathcal{M}|_U = g\mathcal{O}_U \qquad g \in k(X)$$
$$\mathcal{L} \otimes \mathcal{M}|_U = fg\mathcal{O}_U = f\mathcal{O}_U \otimes_{\mathcal{O}_U} g\mathcal{O}_U$$

then $\mathcal{L} \otimes \mathcal{M}$ really is the tensor product as $\mathcal{O}_X$-modules.

1.1. Pullbacks of line bundles.

Suppose we have a morphism $f : X \to Y$, and $\mathcal{L} = \{(U_i, g_{ij})\}$ is a line bundle on Y. We define $f^*\mathcal{L}$ to be the line bundle on X given by

$$f^*\mathcal{L} = \{(f^{-1}(U_i), f^*(g_{ij}))\}$$

where

$$f^* : \mathcal{O}_Y(U_{ij}) \to \mathcal{O}_X(f^{-1}(U_{ij}))$$

is the induced map. Note that the cocycle condition is satisfied because f^* is a ring homomorphism.

Remark 1.1. f^* pulls back global sections of $\mathcal{L}$ to $f^*\mathcal{L}$:

$$f^* : \Gamma(Y, \mathcal{L}) \to \Gamma(X, f^*\mathcal{L})$$
$$s \mapsto f^*s$$

To see this, note that if $s = \{s_i \in \mathcal{O}_Y(U_i)\}$, then $f^*s = \{f^*s_i \in \mathcal{O}_X(f^{-1}(U_i))\}$ and again the compatibility conditions are still satisfied.

Remark 1.2. Note that f^* is a group homomorphism of the Picard groups, since

$$f^*(\mathcal{L} \otimes \mathcal{M}) = f^*\mathcal{L} \otimes f^*\mathcal{M}.$$

Last time on $\mathbf{P}^n$ we defined $\mathcal{O}_{\mathbf{P}^n}(r)$ line bundles using the standard cover of $\mathbf{P}^n$ given by $U_i = \{x_i \neq 0\}$ by

$$g_{ij} = \left(\frac{x_i}{x_j}\right)^r$$

and we noted that

$$\Gamma(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(r)) = \{\text{homogenous polynomials of degree } r \text{ in } x_0, \dots, x_n\}$$

so we have

$$\mathrm{Pic}(\mathbf{P}^n) = \mathbb{Z}[\mathcal{O}_{\mathbf{P}^n}(1)]. \tag{1}$$

There is no torsion here.

Let's do something funny. Let Y be the complement of an irreducible curve in $\mathbf{P}^2$, and $i : Y \hookrightarrow \mathbf{P}^2$ its inclusion. Then we have a pullback homomorphism

$$i^* : \mathrm{Pic}(\mathbf{P}^2) \to \mathrm{Pic}(Y).$$

Suppose Y is the complement of an irreducible conic. Let's figure out $\mathrm{Pic}(Y)$. So $Y = \{[x, y, z] : f_2(x, y, z) \neq 0\}$ where $f_2 \in k[x, y, z]$ is a homogenous polynomial of degree 2. We have the line bundle $\mathcal{O}_Y(2) = i^*\mathcal{O}_{\mathbf{P}^2}(2)$. Note that the transition functions remain the same. We claim that $\mathcal{O}_Y(2) = \mathcal{O}_Y$ (i.e., $\mathcal{O}_Y(2)$ is trivial) but $\mathcal{O}_Y(1) \neq \mathcal{O}_Y$. This implies that $\mathcal{O}_Y(1)$ is a torsion element of order 2 in $\mathrm{Pic}(Y)$. To see these claims, recall that a bundle is trivial if we can produce an everywhere nonzero section. Note that $s = f_2 \in \Gamma(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(2))$, so $s|_Y \in \Gamma(Y, \mathcal{O}_Y(2))$ and $s(y) \neq 0$ for all $y \in Y$, so $\mathcal{O}_Y(2)$ is indeed trivial. To show that $\mathcal{O}_Y(1) \neq \mathcal{O}_Y$ we need to show that any section must vanish somewhere on Y. First, we show that $i^* : \Gamma(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(1)) \xrightarrow{\sim} \Gamma(Y, \mathcal{O}_Y(1))$ is an isomorphism. Then we note that any $s \in \Gamma(\mathcal{O}_{\mathbf{P}^2}(1))$ vanishes along a line in $\mathbf{P}^2$ (linear polynomials), but since Y^c is nondegenerate, no line in $\mathbf{P}^2$ is contained in Y^c, so the zero locus of any section will have to meet Y.

2. MAPS TO PROJECTIVE SPACE

Suppose X is a variety, $L \to X$ a line bundle and $s^0, \ldots, s^n \in \Gamma(X, L)$ are sections which do not vanish simultaneously anywhere. Let

$$\phi : X \to \mathbf{P}^n$$

$$\phi = [s^0, \ldots, s^n].$$

We do we mean by this? Let $L = \{(U_i, g_{ij})\}$. Then $s^\alpha = \{s_i^\alpha \in \mathcal{O}_X(U_i)\}$ for $0 \leq \alpha \leq n$. Then

$$\phi|_{U_i}(x) = [s_i^0(x), \ldots, s_i^\alpha(x), \ldots, s_i^n(x)]$$

and $s_j^\alpha = g_{ij} s_i^\alpha$. We need to check that $\phi|_{U_{ij}}$ makes sense:

$$[s_j^0(x), \ldots, s_j^n(x)] = [g_{ij}(x)s_i^0(x), \ldots, g_{ij}(x)s_i^n(x)] = [s_i^0(x), \ldots, s_i^n(x)]$$

since this is just a rescaling, so the definitions agree on the overlap, and $\phi : X \to \mathbf{P}^n$ is well-defined everywhere, and it is an algebraic map because $\phi|_{U_i}$ is given by regular functions, so it is morphism.

Theorem 2.1. *$\phi^*(\mathcal{O}_{\mathbf{P}^n}(1)) \simeq L$.*

Recall that $\phi^* : \Gamma(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1)) \to \Gamma(X, L)$, where $\Gamma(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1))$ is the set of linear functions in $x^0, \ldots, x^n$. We claiming that $\phi^*(x^\alpha) = s^\alpha$. So given *any* morphism, pulling back to X on $\mathcal{O}_{\mathbf{P}^n}(1)$ gives us the sections which "define" the map.

Proof. Let $V^\alpha = \{x^\alpha \neq 0\}$ be our standard cover of $\mathbf{P}^n$ with open affines, and write $\mathcal{O}_{\mathbf{P}^n}(1) = \{(V^\alpha, g_{\alpha\beta} = x^\alpha/x^\beta)\}$. Then

$$\phi^*\mathcal{O}_{\mathbf{P}^n}(1) = \{U^\alpha = \phi^{-1}(V^\alpha) = \{s^\alpha \neq 0\}, \phi^*(x^\alpha/x^\beta)\}.$$

We claim that this is isomorphic to our initial bundle. Note that $\phi^*(x^\alpha/x^\beta) = s^\alpha/s^\beta \in \mathcal{O}_X^\times(U^\alpha \cap U^\beta)$. For $x \in U_i \cap U^\alpha \cap U^\beta$,

$$\frac{s^\alpha}{s^\beta}(x) = \frac{s_i^\alpha(x)}{s_i^\beta(x)}$$

and this is consistent on overlaps $U_{ij} \cap U^\alpha \cap U^\beta$:

$$\frac{s_j^\alpha(x)}{s_j^\beta(x)} = \frac{g_{ij}(x)s_i^\alpha(x)}{g_{ij}(x)s_i^\beta(x)} = \frac{s_i^\alpha(x)}{s_i^\beta(x)}$$

so it is well-defined.

We now want to construct new trivializations:

$$\begin{tikzcd}
\eta^\alpha : U^\alpha \times \mathbb{A}^1 \arrow[rr] \arrow[dr, "\mathrm{pr}_1"] & & \pi^{-1}(U^\alpha) \arrow[dl, "\pi"] \\
& U^\alpha &
\end{tikzcd}$$

such that the transition functions

$$\begin{tikzcd}
U^{\alpha\beta} \times \mathbb{A}^1 \arrow[rr, "\eta^\alpha|_{U^{\alpha\beta}}"] \arrow[dr, "s^\alpha/s^\beta"'] & & \pi^{-1}(U^{\alpha\beta}) \\
& U^{\alpha\beta} \times \mathbb{A}^1 \arrow[ur, "\eta^\beta|_{U^{\alpha\beta}}"] &
\end{tikzcd}$$

commute. Let $\eta^\alpha(x,\lambda) = \lambda \underbrace{s^\alpha(x)}_{\neq 0}$ and it is a point in the fiber. So if $x \in U_i$, then $\eta^\alpha(x,\lambda) = \phi_i(x, \lambda\, s_i^\alpha(x))$, where ϕ_i is our original trivialization of L. We still need to check that η^α is well-defined on U_{ij}, but this is immediate:

$$\begin{tikzcd}
(x,\lambda) \in U_i \arrow[rr, mapsto, "\eta^\alpha"] \arrow[dr, mapsto, "s^\alpha/s^\beta?"'] & & \phi_i(x, \lambda\, s_i^\alpha(x)) = \phi(x, \mu\, s_i^\beta(x)) \\
& (x,\mu) \arrow[ur, mapsto, "\eta^\beta"] &
\end{tikzcd}$$

Then

$$\mu = \lambda \frac{s_i^\alpha(x)}{s_i^\beta(x)} = \lambda \left(\frac{s^\alpha}{s^\beta}\right)(x)$$

so s^α/s^β really works (and we can remove the "?"). We claim victory:

$$\phi^*\mathcal{O}_{\mathbf{P}^n}(1) \simeq L.$$

□

Proposition 2.2. *Given any map* $\phi : X \to \boldsymbol{P}^n$*, we can create a line bundle* $L \to X$ *by* $L = \phi^*\mathcal{O}_{\boldsymbol{P}^n}(1)$ *with sections* $s^\alpha = \phi^*(x^\alpha)$*,* $0 \leq \alpha \leq n$*, these do not vanish simultaneously anywhere on* X*, and such that* $\phi = [s^0, \dots, s^n]$*.*

So we have an equivalence:

$$\{\phi : X \to \mathbf{P}^n\} \xleftrightarrow{1:1} \begin{Bmatrix} \text{Line bundles } L \to X, \text{ sections } s_0, \dots, s_n \in \Gamma(X, L) \\ \text{not vanishing simultaneously} \end{Bmatrix}$$

Our varieties are abstract, but it's good to embed them in $\mathbf{P}^n$ every now and then (basically all the time).

3. The Picard group of a Curve

Let C be a smooth curve. Define $\operatorname{Div}(C)$ to equal the free abelian group generated by the points of C:

$$\operatorname{Div}(C) = \{n_p \cdot p \mid n_p \in \mathbb{Z}\},$$

so we're only looking at finite sums.

Terminology/Notation 3.1. $D = n_{p_1}p_1 + \cdots + n_{p_s}p_s$ is called a *divisor* of C. If $n_p \geq 0\ \forall p \in C$, then D is an *effective divisor.* If $f \in k(C)^\times\ \forall p \in C$ then $\mathcal{O}_{C,p}$ is a DVR with valuation ord_p: $k(C)^\times \to \mathbb{Z}$. Then we associate to f

$$f \mapsto \operatorname{div}(f) = (f) = \sum_{p \in C} \operatorname{ord}_p(f) \cdot p.$$

Here, (f) is called the *principal divisor.*

How do we know that this is a finite sum?

Remark 3.2. $\forall f \neq 0$ there are only finitely many valuations on $k(C)/k$ such that $v(y) \neq 0$. Then

$$\operatorname{Div}^0(C) = \{\operatorname{div}(f) \mid f \in k^\times(0)\}$$

is a subgroup of $\operatorname{Div}(C)$. It follow from $\operatorname{div}(f)$+$\operatorname{div}(g)$ =$\operatorname{div}(f \cdot g)$ that

$$Cl(C) = \frac{\operatorname{Div}(C)}{\operatorname{Div}^0(C)},$$

where $Cl(C)$ denotes the class group of C. Thus, this turns out to be a covering isomorphism to $\operatorname{Pic}(C)$.

Example 3.3. Consider $C = \mathbf{P}^1$ and let $f(t) = \dfrac{1}{t} \in k(C)$. Then $\operatorname{div}(f) = \infty - 0$. Also, $\operatorname{div}(t) = 0 - \infty$. These ∞'s and 0's are all coming from the formula { number of poles} − {number of zeros}. Thus,

$$\operatorname{div}\left(\frac{t^2 - 2t}{t_3}\right) = (2) + (0) - (3) - (\infty).$$

We then see that $\operatorname{Div}^0(\mathbf{P}^1) = \{\sum n_p \cdot p \mid \sum n_p = 0\}$.

Example 3.4. If $f = \dfrac{(t-a_1)\cdots(t-a_m)}{(t-b_1)\cdots(t-b_n)}$ then

$$\operatorname{div}(f) = a_1 + \cdots + a_m - (b_1 + \cdots + b_n) + (n-m)\infty.$$

Example 3.5. If $D = 2\cdot(1) + 3\cdot(2) - 4\cdot(7) - \cdot(9)$, then

$$f = \frac{(t-1)^2(t-2)^3}{(t-7)^4(t-9)} \in k(\mathbf{P}^1).$$

We also have $Cl(\mathbf{P}^1) = \dfrac{\operatorname{Div}(\mathbf{P}^1)}{\operatorname{Div}^0(\mathbf{P}^1)} \cong \mathbb{Z}$ by the degree map:

$$\deg : \operatorname{Div}(\mathbf{P}^1) \to \mathbb{Z}$$

where $\deg(D) = \sum n_p$.