Algebraic Geometry Notes 4/20- Notes taken by Jason Deblois

Notation Let C be a smooth curve.

- For $D, D' \in \mathrm{Div}(C)$, we say D and D' are *linearly equivalent*, written $D \equiv D'$ or $D \sim D'$, iff $D = D' + div(f)$ for some $f \in k(C)^*$.
- For $D = n_1 p_1 + \ldots n_s p_s$, the *support* of D is $\mathrm{supp}(D) = \{p_1, \ldots, p_s\}$.

Theorem 1. *For C a smooth curve, there is a canonical isomorphism*

$$\mathrm{Cl}(C) \to \mathrm{Pic}(C)$$

Proof. We define a map $\phi : \mathrm{Div}(C) \to \mathrm{Pic}(C)$ with $\mathrm{Div}^0(C) \subseteq \ker(\phi)$, thus inducing a map $\phi : \mathrm{Cl}(C) \to \mathrm{Pic}(C)$.

For $D = \sum n_p \cdot p \in \mathrm{Div}(C)$, define a sheaf $\mathcal{O}_C(D)$ as follows: for $U \subset C$ open,

$$\Gamma(U, \mathcal{O}_C(D)) = \{f \in k(C) : \forall\, p \in U,\ ord_p(f) + n_p \geq 0\} \subseteq k(C)$$

This is clearly a sheaf, in fact an $\mathcal{O}_C$-module, since restrictions are inclusions inside $k(C)$. We show below that this is an invertible sheaf, hence the sheaf of sections of a line bundle.

Example Suppose $D = 0$. Then

$$\begin{aligned} \Gamma(U, \mathcal{O}_C(D)) &= \{f \in k(C) \ : \ ord_p(f) \geq 0 \ \forall\, p \in U\} \\ &= \bigcap_{p \in U} \mathcal{O}_{C,p} \\ &= \mathcal{O}_C(U) \end{aligned}$$

Thus $\mathcal{O}_C(0) = \mathcal{O}_C$.

To show $\mathcal{O}_C(D)$ is an invertible sheaf, we show it is locally isomorphic to $\mathcal{O}_C$. Let $U_0 = C - \mathrm{supp}(D)$. For $U \subset U_0$ open, by definition

$$\Gamma(U, \mathcal{O}_C(D)) = \{f \in k(C) \ : \ ord_p(f) \geq 0 \ \forall\, p \in U\} = \mathcal{O}_C(U).$$

Hence $\mathcal{O}_C(D)|_{U_0} = \mathcal{O}_{U_0}$.

Write $D = n_1 p_1 + \ldots n_s p_s$, and for each i fix $f_i \in k(C)$ such that $ord_{p_i}(f_i) = n_i$ (for example by taking $f = t^{n_i}$, where t is a uniformizing parameter for $\mathcal{O}_{C,p_i}$). Then $div(f_i) = n_i p_i + E_i$, where p_i is not in the support of E_i. Let

$U_i = C - \text{supp}(E_i) - \bigcup_{j \neq i} \{p_j\}$. Now for $U \subset U_i$ open,

$$\begin{aligned}
\Gamma(U, \mathcal{O}_C(D)) &= \{f \in k(C) \ : \ ord_{p_i}(f) + n_i \geq 0 \text{ and } ord_p(f) \geq 0 \ \forall \ p \neq p_i\} \\
&= \{f \in k(C) \ : \ ord_{p_i}(f) + ord_{p_i}(f_i) \geq 0 \text{ and } ord_p(f) \geq 0 \ \forall \ p \neq p_i\} \\
&= \{f \in k(C) \ : \ ord_p(ff_i) \geq 0 \ \forall \ p \in U\} \\
&= \{f \in k(C) \ : \ ff_i \in \mathcal{O}_C(U)\} \\
&= \frac{1}{f_i} \cdot \mathcal{O}_C(U) \subset k(C)
\end{aligned}$$

Thus multiplication by f_i gives an $\mathcal{O}_C(U)$-module isomorphism between $\Gamma(U, \mathcal{O}_C(D))$ and $\mathcal{O}_C(U)$; therefore $\mathcal{O}_C(D)|_{U_i} \cong \mathcal{O}_{U_i}$ as $\mathcal{O}_{U_i}$-modules. This shows $\mathcal{O}_C(D)$ is locally trivial and hence an element of $\text{Pic}(C)$. Define $\phi(D) = \mathcal{O}_C(D) \in \text{Pic}(C)$. By above this gives a map $\text{Div(C)} \to \text{Pic}(C)$.

Now suppose $D = div(u)$ for some $u \in k(C)$. Then for any $U \subset C$ open

$$\begin{aligned}
\Gamma(U, \mathcal{O}_C(D)) &= \{f \in k(C) \ : \ fu \in \mathcal{O}_C(U)\} \\
&= \frac{1}{u} \cdot \mathcal{O}_C(U) \cong \mathcal{O}_C(U)
\end{aligned}$$

Thus $\mathcal{O}_C(D)$ is globally isomorphic to $\mathcal{O}_C$, so $\phi(D) = 0 \in \text{Pic}(C)$. This gives an induced map $\phi : \ \text{CL}(C) \to \text{Pic}(C)$.

We wish to define a map $\eta : \ \text{Pic}(C) \to \text{Cl}(C)$. In order to do so, consider a line bundle $L \to C$ and suppose there exists a nonzero section $s \in \Gamma(C, \mathcal{L})$,

$$s = \{s_i \in \mathcal{O}_C(U_i) \ : \ s_j = s_i g_{ij}\}.$$

Then we may define $Z(s) = \sum_{p \in C} ord_p(s) \cdot p \quad \in \text{Div}(C)$; where $ord_p(s) := ord_p(s_i)$ if $p \in U_i$.

In general we must consider *rational sections* $s : C \to L$,

$$s = \{s_i \in k(C) \ : \ s_j = s_i g_{ij}\}.$$

Let $G = \{(L, s) \ : \ L \in \text{Pic}(C), \ s \text{ a rational section of L}\}$, then we will induce $\eta : \text{Pic}(C) \to \text{Cl}(C)$ via the following diagram:

$$\begin{tikzcd}
G \arrow[r, "Z"] \arrow[d] & \text{Div}(C) \arrow[d] \\
\text{Pic}(C) \arrow[r, "\eta"] & \text{Cl}(C)
\end{tikzcd}$$

For a rational section s, analogously define $Z(s) = \sum_{p \in C} ord_p(s) \cdot p$, where $ord_p(s) = ord_p(s_i)$ if $p \in U_i$. (Note that the map Z depends on the line bundle L, however this is suppressed in the notation.)

NB $Z(s) \neq div(s_i)$, for the g_{ij} may have zeros and poles outside U_{ij}.

Observe that the linear equivalence class of $Z(s)$ does not depend on the choice of rational section s: if t is another rational section, then $\frac{s}{t}$ is a well-defined rational function, for

$$\frac{s_j}{t_j} = \frac{s_i g_{ij}}{t_i g_{ij}} = \frac{s_i}{t_i}$$

Therefore $Z(s) = Z(t) + div(s/t)$. There is thus an induced map η given by $\eta(L) = Z(s) \in \mathrm{Cl}(C)$ for any rational section s of L.

We now show that η and ϕ are isomorphisms by showing that they are inverses of each other. For a divisor $D = n_1 \cdot p_1 + \ldots + n_k \cdot p_k$, recall that $\phi(D) = \mathcal{O}_C(D)$ was defined by

$$\Gamma(U, \mathcal{O}_C(D)) = \{f \in k(C) \ : \ ord_p(f) + n_p \geq 0 \ \forall \ p \in U\}$$

for $U \subset C$ open. Note that the rational sections of $\mathcal{O}_C(D)$ are in one-to-one correspondence with elements of $k(C)$: $f \in k(C)$ determines a section s with $s_0 = f$. Now consider the rational section s with $s_0 = 1 \in k(C)$; then we claim that $Z(s) = D$. Clearly for any point $p \in U_0$, $ord_p(s) = ord_p(1) = 0$. On U_i, $s_i = f_i \cdot 1 = f_i$, where f_i as above is a function so that $ord_{p_i}(f_i) = n_{p_i}$. Thus $ord_{p_i}(s) = ord_{p_i}(s_i) = ord_{p_i}(f_i) = n_{p_i}$, and hence $\eta \circ \phi$ is the identity.

Now consider a line bundle $L = \{U_i, g_{ij}\}$. Define a rational section s by $s_0 = 1$; then for $i > 0$, $s_i = g_{0i} s_0 = g_{0i}$. Then $\phi \circ \eta(L) = \mathcal{O}_C(Z(s))$. For $U \subset C$ open,

$$\begin{aligned}
\Gamma(U, \mathcal{O}_C(Z(s))) &= \{f \in k(C) \ : \ ord_p(f) + ord_p(s) \geq 0 \ \forall \ p \in U\} \\
&= \{f \in k(C) \ : \ ord_p(f) + ord_p(g_{0i}) \geq 0 \ \forall \ p \in U \cap U_i\} \\
&= \{f \in k(C) \ : \ ord_p(f \cdot g_{0i}) \geq 0 \ \forall \ p \in U \cap U_i\} \\
&= \{f \in k(C) \ : \ f \cdot g_{0i} \in \mathcal{O}_C(U \cap U_i)\} \\
&= \Gamma(U, \mathcal{L}).
\end{aligned}$$

Therefore $\phi \circ \eta$ is the identity, and $\mathrm{Cl}(C)$ is isomorphic to $\mathrm{Pic}(C)$.

$\square$