4/25/05- Notes taken by Mark Luxton

Throughout these notes C will always be a smooth, projective curve. We have a well-defined map $deg : Pic(C) \to \mathbb{Z}$ where $deg\mathcal{O}_C(D) = degD$.

Notation: Let $D, D' \in Div(C)$. Then $D \leq D'$ iff $D - D'$ is effective.

1. The k-vector spaces $\Gamma(C, \mathcal{O}_C(D)$

Our ultimate goal is to answer the Riemann-Roch problem, that is, to understand $\Gamma(C, \mathcal{O}_C(D))$ for all $D \in Div(C)$. Since these are all finite dimensional vector spaces over k, we're asking to find the dimension. For example, if $D = 0$, $\Gamma(C, \mathcal{O}_C(D)) = k$.

Lemma 1.1. *If $\mathcal{L}$ is a line bundle and $\Gamma(C, \mathcal{L}) \neq 0$, $deg\mathcal{L} \geq 0$. Moreover, if $deg\mathcal{L} = 0$, $\mathcal{L} \simeq \mathcal{O}_C$.*

Proof. Let $0 \neq s \in \Gamma(C, \mathcal{L})$. Then $Z(s) = \sum_p ord_p(s) \cdot p$ is an effective divisor so $degZ(s) = deg\mathcal{L} \geq 0$. If $deg\mathcal{L} = 0$, $\mathcal{L}$ has a regular, nowhere vanishing global section and so $\mathcal{L} \simeq \mathcal{O}_C$ and $s \in k$. □

Proposition 1.2. 1) *If $D \leq D'$, $\Gamma(C, \mathcal{O}_C(D)) \hookrightarrow \Gamma(C, \mathcal{O}_C(D'))$.*

2) *$dim_k\Gamma(C, \mathcal{O}_C(D)) \leq \infty$. In fact, $dim_k\Gamma(C, \mathcal{O}_C(D)) \leq deg(D) + 1$.*

Proof. 1) is obvious if we view both as living inside $k(C)$, the function field of C.

For 2), by the lemma, we may assume $degD \geq 0$. In this case, $\Gamma(C, \mathcal{O}_C(D)) \neq 0$ since we may take $1 \in k(C)$.

Now write $D = p_1 + \ldots + p_s$ where the p_i's are not necessarily distinct. We then have $0 \hookrightarrow \Gamma(\mathcal{O}_C(p_1) \hookrightarrow \ldots \hookrightarrow \Gamma(\mathcal{O}_C(p_1 + \ldots + p_r)$ so it is enough to show that for all $E \in Div(C), p \in C$, $dim_k \frac{\Gamma(\mathcal{O}_C(E+p))}{\Gamma(\mathcal{O}_C(E))} \leq 1$.

Write $E = ap + E'$ where $p \notin Supp(E')$ and $a < 0$. Fix a uniformizing parameter $t \in \mathcal{O}_{C,p}$ at p. Define $\phi : \Gamma(\mathcal{O}_C(D)) \to k$ by $s \mapsto (st^{a+1})(p) \in k$. Note that $(st^{a+1})(p) \in \mathcal{O}_{C,p} \to \mathcal{O}_{C,p}/\mathfrak{m}_p \simeq k$. But $ker\phi = \{s \in \Gamma(\mathcal{O}_C(D)) \mid ord_ps = a\} = \Gamma(\mathcal{O}_C(E)$ so $\frac{\Gamma(\mathcal{O}_C(E+P)}{\Gamma(\mathcal{O}_C(E)} \hookrightarrow k$. □

2. Sheaf of Differentials

Let X be any variety. We'll construct a sheaf $\Omega^1_{X/k}$ which will be the algebraic analogue of the cotangent bundle from differential geometry. To carry this out, for any k-algebra R we'd like an associated module $\Omega^1_{R/k}$ so that if $U \subset X$ is open and affine, $\Gamma(U, \Omega^1_{X/k}) = \Omega^1_{\mathcal{O}_X(U)/k}$.

2.1. The Module of Differentials.

First some algebra: Let R be a finitely generated k-algebra. A derivation of R is a map $d : R \to M$ where M is an R-module such that:

(1) $$d(x + y) = dx + dy \quad \text{(additive)}$$

(2) $$d(\lambda) = 0 \quad \text{for all } \lambda \in k$$

(3) $$d(xy) = xd(y) + yd(x) \quad \text{(Leibniz Rule)}$$

Theorem 2.1. *There exist an R-module $\Omega^1_{R/k}$ together with a derivation $d : R \to \Omega^1_{R/k}$ satisfying the following universal property: For all derivations $D : R \to M$, there is a unique R-module morphism $\phi : \Omega^1_{R/k} \to M$, such that $D = \phi \circ d$. $\Omega^1_{R/k}$ is called the module of (Kähler) differentials.*

Proof. We simply take $\Omega^1_{R/k}$ to be the free R-module generated by $\{d\alpha\}_{\alpha \in R}$ modulo the three above relations. □

2.1.1. *Example.* If R is finitely generated over k by $x_1, \dots, x_n$, $\Omega^1_{R/k}$ is finitely generated over k with generators $dx_1, \dots, dx_n$. If $f(x_1, \dots, x_n) \in k[x_1, \dots, x_n]$, $df(x_1, \dots, x_n) = \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i \in \Omega^1_{R/k}$.

2.1.2. *Example.* If $R = k[t]$, $\Omega^1_{R/k} = k[t] < dt >$. If $R = k[x_1, \dots, x_n]$, $\Omega^1_{R/k} = \bigoplus_{i=1}^n R \cdot dx_i$, the free R-module generated by $dx_1, \dots, dx_n$. A typical $\omega = \sum a_i(x_1, \dots, x_n) \cdot dx_i$.

2.1.3. *Remark.* $\Omega^1_{R/k}$ behaves nicely with respect to localization: if $S \subset R$ is multiplicative, consider $S^{-1}R$. Then $\Omega^1_{S^{-1}R/k} \simeq S^{-1}(\Omega^1_{R/k})$ canonically.

2.1.4. *Example.* If $X \subset \mathbb{A}^n$, $A(X) = \mathcal{O}_X(X)$, and $\mathfrak{m}_p \subset A(X)$ corresponds to a point $p \in X$, $\mathcal{O}_{X,p} = A(X)_{\mathfrak{m}_p}$, $\Omega^1_{\mathcal{O}_{X,p}/k} = (\Omega^1_{A(X)/k})_{\mathfrak{m}_p}$. This is important for constructing a sheaf as it gives the stalk at each point.

2.2. **Sheaf of Differentials on a Curve.** Let $K = k(C)$. Then $\Omega^1_{K/k}$ is a one-dimensional vector space over K with basis given by dt for any $t \in K \backslash k$: take any $t \in K \backslash k$. Then $[K : k(t)] < \infty$. We would like for any $f \in K$, $df = (rational function) \cdot dt$.

f satisfies some polynomial $a_n(t) f^n + \dots a_1(t) f + a_0(t) = 0$ with $a_i \in k[t]$. Suppose n is minimal. Differentiating we get $g(t, f)dt + h(t, f)df = 0$ for some $g, h \in k[t, f]$, where $h(t, f)$ has degree $n - 1$ in f so that $h \neq 0$. Hence $df \in K \cdot dt$.

We call $\Omega^1_{K/k}$ the space of meromorphic differentials.

2.2.1. *The Divisor of a Meromorphic Differential.* Take any $0 \neq \omega \in \Omega^1_{K/k}$. We will associated to ω the divisor $div(\omega) = \sum_{p \in C} ord_p(\omega)$. We then want to define $ord_p(\omega)$.

Fix $p \in C, t \in \mathcal{O}_{C,p}$ a uniformizing parameter. We've seen that dt generates $\Omega^1_{K/k}$ and so $\omega = f dt$ for some $f \in K$. We therefore define $ord_p(\omega) = ord_p(f)$.

If $\omega, \omega' \in \Omega^1_{K/k} \backslash \{0\}$, $\omega' = f\omega$ for some $f \in K$. So $div(\omega') = div(\omega) + div(f)$ iff $div(\omega') \equiv div(\omega)$. We therefore define the canonical bundle to be $K_C = \Omega^1_C = \mathcal{O}_C(div(\omega))$ for any $\omega \in \Omega^1_{K/k} \backslash \{0\}$.