Algebraic Geometry

1 4/4/05

In this lecture we develop a correspondence between the categories of smooth curves, C, (over an algebraically closed field, k throughout) and finitely generated field extensions, K/k, of transcendence degree 1. Of course, we have not yet defined the term, *curve*, so we should start there.

Definition 1. *A curve is a smooth algebraic variety of dimension* 1.

We have already worked on the forward direction of our correspondence: given a curve C/k, we assign to it the field $K = k(C)$, its function field. Note the assumption that C be a curve includes the hypothesis that this field to have transcendence degree 1. Under this correspondence, a point on the curve is sent to the subring $O_{C,p}$, the local ring at that point, which is a DVR with valuation $ord_p : K^* \to \mathbb{Z}$. Moreover, $O_{C,p} = \{f \in K : ord_p(f) \geq 0\} \cup \{0\}$, the valuation ring for ord_p.

Recall that a Dedekind domain is an integrally closed Noetherian domain of dimension 1, meaning that all non-zero prime ideals are maximal.

Theorem 1.1. *Let R be a domain of dimension* 1*. Then R is Dedkind if and only if, for all $p \in MSpec(R)$, R_p is a DVR.*

Proof. Recall that a local ring R_p is a DVR if and only if it is regular, which is equivalent to its maximal ideal being principal, which is also equivalent to R_p being integrally closed. Thus the theorem reduces to: R is integrally closed if and only if R_p is integrally closed for all $p \in MSpec(R)$. (This is a familiar sort of local to global relationship that holds for several of our favorite properties in commutative algebra. For example, a ring is Noetherian if and only if all of its localizations are Noetherian.) The forward direction of this assertion is a simple check and will be omitted. The reverse direction can be broken down into several easy steps, some of which have been on the homework. Note that

$$R = \bigcap_{p \in MSpec(R)} R_p \subset Q(R).$$

Also, note that the intersection of any collection of integrally closed localization of R is integrally closed. $\square$

Theorem 1.2. *Let R be a Dedekind domain, $Q(R) = K$, and L/K be a finite field extension. Then if B is the integral closure of R in L, B is also Dedekind. Moreover, if R is a finitely generated k-algebra, then B is a finitely generated k-algebra as well.*

The proof is difficult and omitted. This theorem does, however, motivate the definition of a Dedekind domain, which heretofore seemed arbitrary. The theorem tells us that this is a stable property under a crucial procedure of commutative algebra. Also, since $\mathbb{Z}$ is a Dedekind domain, the theorem tells us that its integral closure in any finite extension of $\mathbb{Q}$ is Dedekind as well, a fact of great interest to number theorists. We generally do not consider $\mathbb{Q}$ as a possible field in this course but mention it in passing. Now we recall a theorem from last week:

Theorem 1.3. *Suppose K/k is a finitely generated field extension of transcendence degree 1 and $y \in K$. Then there exist only finitely many valuations of K/k, v, such that $v(y) < 0$.*

We will not reprove this, but we will rekindle our intuition for it. Let $K = k(\mathbb{P}^1) = k(t)$ and $y = \frac{f(t)}{g(t)} \in K$. The valuations for which $v(y) < 0$ correspond to the zeros of g, of which there are certainly only finitely many.

Theorem 1.4 (Fundamental Construction). *Let v be a discrete valuation of K/k. Then there exists a smooth curve C_v and a point $p \in C_v$ such that $K = k(C_v)$ and $v = ord_p$.*

Proof. Denote by (R_v, m_v) the valuation ring for v with maximal ideal m_v. Take an element $y \in m_v$. Consider $k[y] \subseteq K$. This is, as the notation suggests, a polynomial ring because the element y cannot satisfy any polynomials over the algebraically closed field k. Then $k[y] \subseteq k(y) \subseteq K$, and $k[y]$ is finitely genreated over k (as an algebra) and Dedekind being polynomials in a single variable over an algebraically closed field. Because K/k is of transcendence degree 1, $[K : k(y)] < \infty$. Let B_y be the integral closure of $k[y]$ in K. By Theorem **??**, B_y is then a Dedekind domain and a finitely generated k-algebra. Any such ring is the affine coordinate ring of a variety, C_v. Any localization at a maximal ideal of B_y then produces a DVR, which is regular, hence C_v is smooth. It is dimension 1, so C_v is a curve.

Note that $k[y] \subseteq R_v$, and the latter is integrally closed, so $B_y \subseteq R_v$. Let $n = m_v \cap B_y$, which is then a prime ideal of B_y. Note that $y \in n$, so $n \neq (0)$, and thus n is maximal. Also note that B_y and R_v have the same field of fractions, though this is non-trivial. Then $(R_v, m_v) = ((B_y)_n, n(B_y)_n) = (O_{C_v,p}, m_p)$, and we have found p. □

Note that R, or really v, did not play a crucial role in this construction until the very end, and we claim to have actually constructed a curve such that all but finitely many of the valuations on K, the ones with $v(y) \geq 0$, correspond to the order of vanishing of some point $q \in C$.

Scholium 1.5. *Let C_v be as in the previous theorem. Then all valuations w of K/k such that $w(y) \geq 0$ are realized as ord_q for some point $q \in C_v$.*

Proof. Let w be a valuation on K/k such that $w(y) \geq 0$ with corresponding valuation ring (R_w, m_w). Because $w(y) \geq 0$, we have that $y \in R_w$. If $y \in m_w$, then we can use B_y to construct a curve for w. Then $C_w = C_v$ and we are done. If $w(y) = 0$, then denote by $\tilde{c}$ the class of y in $k = R_w/m_w$ and c a lift of $\tilde{c}$. Then let $y' = y - c \in m_w$. Then $k[y] = k[y']$ as subrings of K. This reduces us to the previous case, because $B_{y'} = B_y$ again. □

Hence by Theorem **??** for any extension K/k we can get a finite collection of curves such that every valuation on the field corresponds to the order of vanishing at (exactly, as we'll see later) one point on (each of at least) one of the curves. We will not have time to complete the construction, but we will proceed from this point to glue these curves together into one projective curve where there is a bijection between valuations and points.

Recall:

Lemma 1.6 (Algebraic L'Hopital). *Suppose C is a curve and $\phi : C \dashrightarrow Y$ is a rational map to a projective variety. Then there exists a unique extension $\tilde{\phi} : C \to Y$.*

Using this result, we can prove that:

Theorem 1.7. *Every smooth curve is quasi-projective, namely open in a projective curve.*

Proof. Note that C can be covered by affines, $C_i \subset \mathbb{A}^{n_i} \subset \mathbb{P}^{n_i}$. Let Y_i be the closure of C_i in $\mathbb{P}^{n_i}$ and $j_i : C_i \hookrightarrow Y_i$ be the embeddings. Then the j_i are rational maps on C into projective spaces, so by Theorem **??** there exist unique extensions of the j_i to $f_i : C \to Y_i$. The f_i are not in general embeddings any longer. However, we can let

$$f = (f_i) : C \to \prod Y_i \subset \prod \mathbb{P}^{n_i} \overset{Segre}{\hookrightarrow} \mathbb{P}^N.$$

Let $Y = \overline{f(C)}$. For simplicity, we will assume $C = C_1 \cup C_2$. Then $f|_{C_1}(y) = (y, f_2(y))$ and $f|_{C_2}(y) = (f_1(y), y)$. Then, locally, f^{-1} is just the projection onto one of its coordinates, which is a local isomorphism. Suppose $f(x) = f(y) = z$. Then f^* sends $O_{Y,z}$ to $O_{C,x}$ and $O_{C,y}$ as subrings of $k(C)$. But such a subring determines the point in C. Hence $x = y$, so f is injective. Thus f is an isomorphism of C with its image, which is open in its closure in $\mathbb{P}^N$, making C quasi-projective. There are many more details on this in the next lecture. □

Corollary 1.8. *Any smooth curve is birational to a projective curve.*

Proof. By the previous theorem, we can assume C is embedded in $\mathbb{P}^N$. Then its closure, $\overline{C}$ is a projective curve, and $\overline{C} \setminus C$ is a finite collection of points, so the inclusion is the birational isomorphism. □

Again, our next goal will be to glue the curves with points corresponding to valuations on a field in such a way that there is a bijection between the valuations and points.