

Algebraic Geometry

1. 4/6/2005

(Notes taken by Bill Kalahurka)

Theorem 1.1. ($k = \bar{k}$) *There is an equivalence of categories:*

- (1) *{finitely generated fields \mathbf{K}/k of transcendence degree 1}*
- (2) *{smooth projective curves (i.e. 1-dimensional separated varieties) C/k }*

Given by:

(2) \Rightarrow (1) $C/k \Rightarrow \mathbf{K} = k(C)$

(1) \Rightarrow (2) $\mathbf{K} \rightsquigarrow C_k$: *abstract curve with points corresponding to valuations of \mathbf{K}/k*

N.B. Curve = abstract one dimensional variety (separated).

Theorem 1.2. *Every smooth curve is quasi-projective.*

Proof: Write

$$C = \bigcup_i C_i \text{ where } C_i \subseteq \mathbb{A}^{n_i} \subseteq \mathbb{P}^{n_i}$$

For each inclusion map $j_i : C_i \hookrightarrow \mathbb{P}^{n_i}$ there exists an extension $f_i : C \hookrightarrow \mathbb{P}^{n_i}$

Examine the diagonal map

$$f = (f_i) : C \longrightarrow \prod_i \mathbb{P}^{n_i}$$

View \mathbb{P}^{n_i} embedded in \mathbb{P}^N for some N , via the Segre embedding, and define $Y \equiv f(C) \subseteq \mathbb{P}^N$. Then $f : C \longrightarrow Y$ is a local isomorphism. Now suppose there exist $x, y \in C$ s.t. $f(x) = f(y) = z$ where $x \in C_i, y \in C_j$ with $i \neq j$. Then there are isomorphisms of local rings

$$\mathcal{O}_{Y,z} \xrightarrow{f_x^\#} \mathcal{O}_{C,x} \subseteq k(C) = \mathbf{K}$$

$$\mathcal{O}_{Y,z} \xrightarrow{f_y^\#} \mathcal{O}_{C,y} \subseteq k(C) = \mathbf{K}$$

This implies $\mathcal{O}_{C,x} = \mathcal{O}_{C,y}$ (as subrings of \mathbf{K}). This is not possible.

Theorem 1.3. *Let X be a prevariety and let $x, y \in X$ be contained in the same open affine. If $\mathcal{O}_{X,x} \subseteq \mathcal{O}_{X,y}$ (as subrings of $k(X)$) then $x = y$.*

Proof: Let $x, y \in U \subseteq X$, U -affine, $R = \mathcal{O}_X(U)$. $x \rightsquigarrow \mathfrak{m}, x \rightsquigarrow \mathfrak{n}$, where $\mathfrak{m}, \mathfrak{n}$ are maximal ideals of R . So, $\mathcal{O}_{X,r} \simeq R_{\mathfrak{m}}$ and $\mathcal{O}_{Y,r} \simeq R_{\mathfrak{n}}$. And, $R_{\mathfrak{m}} \subseteq R_{\mathfrak{n}} \iff \mathfrak{m} \subseteq \mathfrak{n} \implies \mathfrak{m} = \mathfrak{n}$.

Remark 1.4. This theorem is true for arbitrary $x, y \in X$ if X is a variety.

Lemma 1.5. *Let C_1, C_2 be smooth curves, and $\alpha : C_1 \longrightarrow C_2$ a birational map. Then one can "glue" C_1 and C_2 , i.e. there exists a smooth curve C , such that C is the union of two sets U_1 and U_2 which are both open subsets of C and $U_1 \simeq C_1, U_2 \simeq C_2$.*

Proof of lemma: Let $C_1 \subseteq \mathbb{P}^{n_1}$, $C_2 \subseteq \mathbb{P}^{n_2}$. Let α be an isomorphism between V_1 and V_2 which are open subsets of C_1 and C_2 , respectively. The map from V_1 into $\mathbb{P}^{n_1} \times \mathbb{P}^{n_2}$ defined by $\nu \mapsto (\nu, \alpha(\nu))$ yields a unique extension $\varphi : C_1 \rightarrow \mathbb{P}^{n_1} \times \mathbb{P}^{n_2}$. Similarly, the map from V_2 into $\mathbb{P}^{n_1} \times \mathbb{P}^{n_2}$ defined by $\mu \mapsto (\alpha^{-1}(\mu), \mu)$ yields a unique extension $\psi : C_2 \rightarrow \mathbb{P}^{n_1} \times \mathbb{P}^{n_2}$. By letting $C = \varphi(C_1) \cup \psi(C_2)$, the lemma is proven.

Proof of Main Theorem: We work with the cofinite topology of valuations ν on \mathbf{K}/k . We have that

$$\mathcal{O}_{C_{\mathbf{K}}}(U) = \bigcup_{\nu} R_{\nu} \subseteq \mathbf{K}$$

$\mathbf{K}/k \rightsquigarrow C_{\mathbf{K}}$. We want that $C_{\mathbf{K}}$ is a projective curve. For ν in $C_{\mathbf{K}}$, there exists a smooth affine curve C_1 , with $k(C_1) = \mathbf{K}$, such that $\nu = \text{ord}_{p_1}$, where $p_1 \in C_1$. Moreover, all but finitely many valuations of \mathbf{K}/k are of type ord_q , where $q \in C_1$. There exist smooth affine curves C_1, \dots, C_n with $k(C_i) = \mathbf{K}$ such that for all $\nu \in C_{\mathbf{K}}$, ν is accounted for by some point $p \in C_i$. So, apply the lemma and glue C_1, \dots, C_n to get a smooth curve C such that for all $\nu \in C_{\mathbf{K}}$, there exists a unique $p \in C$ such that $\nu = \text{ord}_p$.

Claim: C is projective

Proof: Suppose C is not projective, so C is quasi-projective, $C \subseteq \mathbb{P}^m$. Take $\overline{C} \in \mathbb{P}^m$, $p \in \overline{C} - C$. Suppose $p \in \overline{C}_{\text{reg}}$. Then $\mathcal{O}_{\overline{C}, p}$ is a DVR. This implies that we get a new valuation of $\mathbf{K} = k(\overline{C})$ which does not correspond to any point of C , *contradiction*. So let $p \in \overline{C}_{\text{sing}}$. Suppose I can construct a surjective birational map $\partial : \tilde{C} \rightarrow \overline{C}$, where \tilde{C} is smooth. Thus, for any $p \in C$, there exists \tilde{p} with $\partial(\tilde{p}) = p$. So, we get $\mathcal{O}_{\tilde{C}, \tilde{p}} \subseteq k(\tilde{C}) = k(C) = \mathbf{K}$, where $\mathcal{O}_{\tilde{C}, \tilde{p}}$ is a DVR. Thus, we get a new valuation that is not of type $\mathcal{O}_{C, p}$ for some $p \in C$, *contradiction*. This completes the proof of the claim.

Normalization of curves:

Let C be a curve, $p \in C$, with $p \in U \subseteq C$, where U is affine, $A(U) \subseteq \mathbf{K} = k(C)$. Let $\partial : \tilde{C} \rightarrow C$ be an isomorphism of $f = \{f \in k \text{ s.t. } f \text{ satisfies a monic polynomial relation with coefficients in } A(U)\}$, (\tilde{C} - smooth). Define $\widetilde{A(U)} \equiv$ the integral closure of $A(U)$ inside \mathbf{K} .

$$\begin{array}{ccc} A(U) & \xrightarrow{\subseteq} & \widetilde{A(U)} \\ \downarrow & & \downarrow \\ U & \xleftarrow{\partial} & \tilde{U} \end{array}$$

$\widetilde{A(U)} \rightarrow$ finitely generated k -algebra, Dedekind domain. Thus, \tilde{U} is a smooth curve.

Claim $\partial : \tilde{U} \rightarrow U$ is surjective

Proof: $p \in U \Leftrightarrow$ there exists \tilde{p} such that $\partial(\tilde{p}) = p \Leftrightarrow$ for all maximal

ideals \mathfrak{m} of $A(U)$, there exists a maximal ideal $\tilde{\mathfrak{m}}$ of $\widetilde{A(U)}$ such that $\tilde{\mathfrak{m}} \cap A(U) = \mathfrak{m} \Leftrightarrow \mathfrak{m}\widetilde{A(U)} \neq \widetilde{A(U)} \Leftrightarrow$ by Nakayama's lemma (see below), for all $\lambda \in 1 + \mathfrak{m}$, $\lambda\widetilde{A(U)} \neq 0 \Leftrightarrow 1 \notin \mathfrak{m} \Leftrightarrow \mathfrak{m}$ is proper. This completes the proof of the claim.

Lemma 1.6 (Nakayama's lemma). *Let A be a ring, M a finitely generated A -module, and $\mathfrak{a} \subseteq A$ an ideal. If for all $\lambda \in 1 + \mathfrak{a}$, $\lambda M \neq 0$, then $\mathfrak{a}M \neq M$.*

Proof: Let $M = \langle m_1, \dots, m_n \rangle$. Assume to the contrary $\mathfrak{a}M = M$, then there exist $a_{ij} \in \mathfrak{a}$, $1 \leq i, j \leq n$, so that

$$m_i = a_{i1}m_1 + \dots + a_{in}m_n$$

Thus,

$$\begin{aligned} (a_{11} - 1)m_1 + a_{12}m_2 + \dots + a_{1n}m_n &= 0 \\ a_{21}m_1 + (a_{22} - 1)m_2 + \dots + a_{2n}m_n &= 0 \\ &\vdots \\ a_{n1}m_1 + a_{n2}m_2 + \dots + (a_{nn} - 1)m_n &= 0 \end{aligned}$$

So, notice that $d = \det(a_{ij} - \delta_{ij}) \in 1 + \mathfrak{a}$. Thus, by Cramer's Rule $dm_i = 0$ for all $1 \leq i \leq n$. Therefore $dM = 0$, *contradiction*