Algebraic Geometry Notes 2/14/05-taken by Craig Michoski

Proposition. *Let* X *and* Y *be prevarieties with* $f : X \to Y$ *any map. If* $\exists \{V_i\}$ *a finite affine covering of* Y *and* $\{U_i\}$ *an affine open covering of* X *such that* $f(U_i) \subseteq V_i$ *and* $f^* : \mathcal{O}_Y(V_i) \to \mathcal{O}_X(U_i)$ *for all* i*, then* f *is a morphism.*

Proof: Given last time.

Thus, given two prevarieties X, Y, and a continuous map $f : X \to Y$ with $(X, \mathcal{O}_X)$, to chech that f is an algebraic morphism it suffices to find open affine coverings $\{U_i\} \subseteq X$ and $\{V_i\} \subseteq Y$, with $f(U_i) \subseteq V_i$ and such that $f^* : \mathcal{O}_Y(V_i) \to \mathcal{O}_X(U_i)$ for all i, that is, f pulls back regular functions on V_i to regular functions on U_i.

Example: Take $\pi : \mathbb{P}^2 - \{[0,0,1]\} \to \mathbb{P}^1$. Then

$$\pi[x,y,z] = [x,y]$$

and since $\mathbb{P}^2 = \mathbb{A}^2 \cup (\mathbb{P}^1)_\infty$ then $\mathbb{A}^2 \hookrightarrow \mathbb{P}^2$ with $(x,y) \overset{\infty}{\mapsto} [x,y,1]$. Since $\mathbb{P}^1 \hookrightarrow \mathbb{P}^2$ and we define $\mathbb{P}^1_\infty = \{[x,y,0] : [x,y] \in \mathbb{P}^1\}$, then we are mapping from the origin to the line at infinity.

To check that π is a morphism, find an affine covering of $\mathbb{P}^1 = V_0 \cup V_1$ with coordinates $\{[u,v]\}$ and

$$V_0 = \{u \neq 0\} \subseteq \mathbb{P}^1$$

$$V_1 = \{v \neq 0\} \subseteq \mathbb{P}^1.$$

As $V_0 \simeq \mathbb{A}^1$ implies $[1,t] \mapsto t$ and $V_1 \simeq \mathbb{A}^1$ implies $t \mapsto [t,1]$, we take

$$\pi^{-1}(V_0) = U_x = \{[x,y,z] : x \neq 0\} \simeq \mathbb{A}^2$$

and

$$\pi^{-1}(V_1) = U_y = \{[x,y,z] : y \neq 0\} \simeq \mathbb{A}^2,$$

so for $U_x \simeq \mathbb{A}^2$ and $V_0 \simeq \mathbb{A}^2$ the map in charts implies that $\mathbb{A}^2 \to \mathbb{A}^1$ and $U_x \xrightarrow{\pi} V_0$. This leads to an algebraic morphism of affine varieties as $[1,a,b] \overset{\pi}{\mapsto} [1,a]$, $[1,a] \mapsto [a]$, $[a,b] \mapsto [a]$ and $(a,b) \mapsto [a]$.

Now you can check it for the isomorphism $U_y \to V_1$ by again getting $(a,b) \mapsto a$ and recalling that $U_x \cup U_y = \mathbb{P}^2 - \{[0,0,1]\}$.

Example: Recall as motivation that complex projective varieties are compact in the Euclidean topology. Consider the projective closure of the affine curve $X_0 : \{y = x^2\} \subseteq \mathbb{A}^2$ again with $\mathbb{A}^2 \cup (\mathbb{P}^1)_\infty \subseteq \mathbb{P}^2$ and an inclusion $i : \mathbb{A}^2 \hookrightarrow \mathbb{P}^2$ so that

$i(x, y) = [x, y, 1]$. Take the projective closure of the parabola X_0 in $\mathbb{P}^2$ as $\overline{i(X_0)} = X$. We have that $x \mapsto [x, x^2, 1]$ and for the change of coordinates $[u, v, w]$ this implies that $u^2 = vw$ and so $X = \{[u, v, w] \in \mathbb{P}^2 : u^2 = vw\}$. The parabola intersects the line at infinity in a point with multiplicity two. Generally for $S = k[x_0 \dots x_n]$ with $S_d = \{f$ the set of homogeneous polynomials of deg$d\}$ we have $S = \bigoplus_{d \geq 0} S_d$.

We would like to extend the correspondence between affine algebraic sets and ideals in polynomial rings to the case of projective space:

Definition: A homogeneous ideal in $S = k[x_0, \dots, x_n]$ is an ideal $\mathfrak{a}$ generated by homogeneous polynomials.

Remark: $\mathfrak{a} \subseteq S$ homogeneous $\Leftrightarrow \mathfrak{a} = \bigoplus_{d \geq 0} (\mathfrak{a} \cap S_d)$. We associate to any homogeneous ideal $\mathfrak{a} \subseteq S$, the set $Z(\mathfrak{a}) = \{p \in \mathbb{P}^n : f(p) = 0 \quad \forall \quad f$ homogeneous in $\mathfrak{a}\} \subseteq \mathbb{P}^n$.

Theorem. *The Zariski closed sets in $\mathbb{P}^n$ (defined a prevariety by the gluing of $n + 1$ open affine pieces) are precisely the sets of the form $Z(\mathfrak{a})$ with $\mathfrak{a} \subset S$ being a homogeneous ideal.*

Proof: Recall $\mathbb{P}^n = U_0 \cup \dots \cup U_m$ for open affine $U_i \simeq \mathbb{A}^m$ where $U_i = \{p = [x_0 \dots x_n] : x_i \neq 0\}$. $X = Z(\mathfrak{a})$ closed and $\mathfrak{a} \subseteq S$ homogeneous. Then $X = Z(\mathfrak{a})$ closed $\Leftrightarrow \forall i \quad X \cap U_i$ is closed in U_i. Check for $i = 0$, g a homogeneous polynomial. $X \cap U_0 = \{p = [1, x_1 \dots x_n] : \forall g \in \mathfrak{a}, g(1, x_1 \dots x_n) = 0\}$. Let $\tilde{g} = \{g(1, x, \dots x_n = 0)\}$ with $\tilde{g} \in k[y, \dots, y_n$. Then $X \cup U_0 = Z(\langle \tilde{g} : g \in \mathfrak{a} \rangle)$ is closed in U_0. Conversely, say $X \subseteq \mathbb{P}^n$ is closed, and assume that X is irreducible by the Noetherian condition. Let $X \neq 0$ and $X \cap U_0 \neq \emptyset$. Since $U_0 \cap X$ is closed in U_0 we have that $U_0 \cap X = Z(\tilde{\mathfrak{a}})$ where $\tilde{\mathfrak{a}}$ is an ideal in $k[y_1 \dots y_n]$. Now, homogenize every polynomial in $\tilde{\mathfrak{a}}$ so that $g(y_1 \dots y_n) \to G(x_0 \cdots x_n) = x_0^d g\left(\frac{x_1}{x_0} \cdots \frac{x_n}{x_0}\right)$. For example, with the given polynomial

$$y_2^2 + 2y_2 \to \left[\left(\frac{x_1}{x_0}\right)^2 + \frac{2x_2}{x_0}\right] x_0^2 = x_1^2 + 2x_2 x_0.$$

Claim: $Z(\tilde{\mathfrak{a}}) = X$ with $\tilde{\mathfrak{a}}$ the ideal in $k[x_0 \cdots x_n]$ generated by all G's.

It is clear that $X \cap V_0 \in Z(\mathfrak{a})$ by all the G's, so we must have that $X \subseteq Z(\mathfrak{a})$. To prove the reverse take $X = \bigcup_{i=0}^n (X \cap U_i)$ which implies that $X = Z(\mathfrak{a})$.

Projective Nullstellensatz: We have associated to each homogeneous ideal $\mathfrak{a} \subset k[x_0, \dots, x_n]$ the algebraic set $Z(\mathfrak{a}) \subseteq \mathbb{P}^n$. Conversely, to each set $X \subseteq \mathbb{P}^n$, we associate the ideal $I(X)$ the ideal generated by all homogeneous polynomials $f \in k[x_0 \cdots x_n]$ such that $f_{|X} = 0$. Then there is a 1 : 1 correspondence given by $\{$ closed algebraic sets in $\mathbb{P}^n\} \overset{Z}{\leftrightarrows} \{\mathfrak{a}$ radical homogeneous ideals in $k[x_0 \cdots x_n]$ which are different from $(x_0 \cdots x_n)\}$.

Note: $I(\emptyset) = S$. Note also that $(x_0 \ldots x_n)$ is not the ideal of any algebraic set $X \subseteq \mathbb{P}^n$.

Homogeneous Coordinate Ring: For an algebraic set $X \subseteq \mathbb{P}^n$, if $I(X)$ is the corresponding homogeneous ideal, then the homogeneous coordinate ring is defined by $S(X) := S/T(X)$.

Remark: $S(X)$ unlike the affine coordinate ring of an affine variety, does not determine the variety. For instance taking $X \subseteq \mathbb{P}^n$ and $Y \subseteq \mathbb{P}^m$, an isomorphism between between X and Y does not imply that $S(X)$ and $S(Y)$ are isomorphic.

Example: Take $X = \mathbb{P}^1$ and the conic $Y = \{x^2 + y^2 = z^2\} \subseteq \mathbb{P}^2$. For some point $p \in Y$ consider the projection $\pi_p : Y \to X$ as an isomorphism. We have that $S(X) = k[u, v]$ and $S(Y) = k[x, y, z]/(x^2 + y^2 - z^2)$. If we look at the k-vector space given by $S(X)_1 = ku \oplus kv$ and $S(Y)_1 = kx \oplus ky \oplus kz$ we notice the dimensions are different, so $S(X)$ is not isomoprphis to $S(Y)$ although X is isomorphic to Y.