Algebraic Geometry

1. 2/17/2005

(Notes taken by Bill Kalahurka)

Last Time:

$$\varphi : \mathbb{P}^n \times \mathbb{P}^m \hookrightarrow \mathbb{P}^{mn+m+n} \text{(Segre embedding)}$$

Define:

$$Z = Im(\varphi) = \{[z_{ij}]_{0\leq i\leq n, 0\leq j\leq m} : z_{ij}z_{lk} - z_{ik}z_{lj} = 0\}$$

We showed φ is bijective onto Z, and Z is a projective subvariety. Mumford shows that φ is a morphism on the patches. Therefore, φ is an isomorphism between $\mathbb{P}^n \times \mathbb{P}^m$ and Z. Since $\mathbb{P}^n \times \mathbb{P}^m$ is irreducible, so is Z.

Examine $\varphi : \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^3$ where:

$$\varphi([s,t],[u,v]) = [su, sv, tu, tv]$$

If we call the target variables x,y,z, and w respectively. Then we have that $\mathbb{P}^1 \times \mathbb{P}^1$ is isomorphic to the quadric surface $xw = yz$ in $\mathbb{P}^3$. Notice that $x^2 + y^2 + z^2 + w^2 = 0$ goes to $xw = yz$ via a change of variable.

Let Q be a quadric surface. Q has two families of lines (the two rulings of the quadric). To see this, fix $p \in \mathbb{P}^1$, $p = [a_0, b_0]$. Let

$$l_p = \varphi(p \times \mathbb{P}^1) = \{[a_0u, a_0v, b_0u, b_0v] : [u,v] \in \mathbb{P}^1\} \subseteq Q$$

Then, $l_p = (b_0x = a_0z) \cap (b_0y = a_0w) \Rightarrow l_p$ is a line. Similarly, for $q \in \mathbb{P}^1$, let $m_q = \varphi(\mathbb{P}^1 \times \{q\}) \subseteq Q$. Then $\{l_p\}_{p\in\mathbb{P}^1}$ and $\{m_q\}_{q\in\mathbb{P}^1}$ are the two rulings. We have that $l_p \cap l_{p'} = \emptyset$ for $p \neq p'$, and $m_q \cap m_{q'} = \emptyset$ for $q \neq q'$. Also, $l_p \cap m_q = \{\varphi(p,q)\}$

Definition 1.1. A prevariety $(X, \mathcal{O}_X)$ is a variety if for all morphisms $f, g : Y \longrightarrow X$, the locus

$$\{y \in Y : f(y) = g(y)\} \quad (*)$$

is closed in Y.

Theorem 1.2. *A prevariety X is a variety if and only if the diagonal $\Delta \subseteq X \times X$ is closed.*

Proof:($\Rightarrow$) Examine the morphisms $p_1, p_2 : X \times X \longrightarrow X$ where p_1 is projection to the first copy of X and p_2 is projection to the second. Then,

$$\Delta = \{(x,y) \in X \times X : p_1(x,y) = p_2(x,y)\}$$

is closed by the definition of a variety.

($\Leftarrow$) Suppose $\Delta \subseteq X \times X$ is closed. Let $f, g : Y \longrightarrow X$ be morphisms, and let ϕ be the morphism $\phi = (f,g) : Y \longrightarrow X \times X$. In particular, ϕ is continuous. Thus,

$$\{y \in Y : f(y) = g(y)\} = \phi^{-1}(\Delta)$$

And, since Δ is closed, $\phi^{-1}(\Delta)$ is closed. This completes the proof.

Remark 1.3. A topological space X is Hausdorff if and only if $\Delta \subseteq X \times X$ is closed (with product topology on $X \times X$).

Remark 1.4. (*) is the analog of the Hausdorff condition in topology.

A variety is a prevariety that satisfies the *separated-ness* condition (*).

The following table describes the difference in terminology between Mumford and Hartshorne:

Mumford	Hartshorne
prevariety	variety
variety	separated variety
prescheme	scheme
scheme	separated scheme

Remark 1.5. Any subprevariety of a variety is a variety. If X is a variety and $Y \subseteq X$ is locally closed, then Y is a variety (with the induced structure).

Remark 1.6. Products of varieties are varieties.

Proposition 1.7. *An affine variety is a variety.*

Proof: Let X be an affine variety $(X \subseteq \mathbb{A}^n, \mathcal{O}_X)$. Let $f, g : Y \longrightarrow X$ be morphisms. Fix $y \in Y$. Then $f(y) = g(y)$ if and only if for all $\eta \in A(X)$, $\eta(f(y)) = \eta(g(y))$. So,

$$\begin{aligned} \{y \in Y : f(y) = g(y)\} &= \{y \in Y : (f^*(\eta) - g^*(\eta))(y) = 0\} \\ &= Z(< f^*\eta - g^*\eta : \eta \in A(X) >) \end{aligned}$$

which is visibly closed. This completes the proof.

Remark 1.8. Let X be the "$\mathbb{A}^1$ with 2 origins" prevariety, i.e. let U, V be two copies of $\mathbb{A}^1$, and let

$$X = (U \coprod V)/(u \sim u \text{ for } u \neq 0)$$

Then X is not a prevariety. We can see this, by examining the following two morphisms $i, j : \mathbb{A}^1 \longrightarrow X$: For i, identify $\mathbb{A}^1$ with U, and let i be the inclusion map. For j, identify $\mathbb{A}^1$ with V, and let j be the inclusion map. Then,

$$\{x \in \mathbb{A}^1 : i(x) = j(x)\} = \mathbb{A}^1 - \{0\}$$

is not closed.

Proposition 1.9. *Let X be a prevariety. If for all $x, y \in X$ there is an open affine $U \subseteq X$ such that $x, y \in U$, then X is a variety.*

Proof: Let $f, g : Y \longrightarrow X$ be morphisms. Let $Z = \{y \in Y : f(y) = g(y)\}$ We want to show Z is closed. Let $z \in \overline{Z}$, so there exists an open affine $V \subseteq X$ such that $f(z), g(z) \in V$. Let $z \in U \equiv f^{-1}(V) \cap g^{-1}(V)$ (U is clearly open). Now, we have morphisms:

$$f|_U : U \longrightarrow V \qquad g|_U : U \longrightarrow V$$

Since V satisfies the Hausdorff axiom, $Z \cap U = \{y \in U : f(y) = g(y)\}$ is closed in $U \Leftrightarrow \overline{Z} \cap U = Z \cap U \Rightarrow z \in Z \Rightarrow Z$ is closed.

Proposition 1.10. $\mathbb{P}^n$ *is a prevariety.*

Proof: Let $x, y \in \mathbb{P}^n$, then there exists a hyperplane $H \subseteq \mathbb{P}^n$ such that $x, y \notin H$. So, $x, y \notin \mathbb{P}^n - H \simeq \mathbb{A}^n$. Now we can apply the previous proposition.

Corollary 1.11. *Any quasiprojective prevariety (locally closed in some* $\mathbb{P}^n$*) is a variety.*

Proposition 1.12 (Mumford page 39)**.** *If* X *is a variety and* $U, V \subseteq X$ *are open affines, then* $U \cap V$ *is also affine and* $\mathcal{O}_X(U \cap V) = \mathcal{O}_X U \cdot \mathcal{O}_X V$ *(inside* $k(X)$*).*

Remark 1.13. This proposition is not a criterion for a prevariety to be a variety

Example 1.14. Let X be $\mathbb{A}^1$ with 2 origins, as before with$X = (U \coprod V)/(u \sim u$ for $u \neq 0)$. Then $U \cap V = \mathbb{A}^1 - \{0\}$ which is affine. $\mathbb{A}^1 - \{0\} \simeq \{xy = 1\}$

Example 1.15. Now take Y to be $\mathbb{A}^2$ with 2 origins. So, let $U = \mathbb{A}^2$,$V = \mathbb{A}^2$. Then

$$Y = X = (U \coprod V)/(u \sim u \text{ for } u \neq (0,0))$$

is a prevariety, but $U \cap V = \mathbb{A}^2 - (0,0)$ is not affine.