Algebraic Geometry- February 25, 2005, Notes taken by Brian Katz

1 2/25/05

1.1 Varieties

Proposition 1.1. *Let X be a variety with open affine subsets U and V. Then $U \cap V$ is affine and*

$$O_X(U \cap V) = O_X(U) \cdot O_X(V) \subset k(X).$$

This is the compositum, the smallest subring of $k(X)$ containing both $O_X(U)$ and $O_X(V)$.

Remark 1.2. *This is not another criterion for determining whether something is a variety. For example, in the line with doubled origin, the intersection of the two glued affine pieces is $\mathbb{A}^1 \setminus \{0\}$, which is affine, being isomorphic to $xy - 1 = 0$ in $\mathbb{A}^2$. However, the result can be used to show that prevarieties are not varieties. Forexample, if we generalize our cunstruction to $\mathbb{A}^2$ with a doubled origin, then the intersection is $\mathbb{A}^2 \setminus \{0\}$, which is not affine. Therefore this prevariety is not a variety.*

1.2 Rational Maps

Let X be a variety, Y a prevariety, and U an open subset of Y. If $f_U : U \to X$ is an algebraic morphism, can we extend it to a morphism $f : Y \to X$?

Last time, we saw that the projective closure of $Y : y = x^2$ was isomorphic to $\mathbb{P}^1$ via the projection map from a point. Stated carefully, $\pi_p : Y \setminus \{p\} \to \mathbb{P}^1$ via projection from p can be extended to an isomorphism on Y. Was our extension unique? We know that, since $\mathbb{P}^1$ is a variety, any two extensions will agree on a closed subset. But they already agreed on a subset of Y whose closure is Y. Hence in this case the extension is unique. We could extend it because Y is a curve, and we can in general extend maps on such subsets curves.

Now consider the map $\phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ defined by $\phi([x : y : z]) = [yz : xz : xy]$. It is well defined on the open subset $\mathbb{P}^2 \setminus \{[0 : 0 : 1], [0 : 1 : 0], [1 : 0 : 0]\}$. This function cannot be extended to all of $\mathbb{P}^2$. In the previous example, we took the limiting process to our projection, the tangent direction at p, to extend the map π_p to all of Y. In this example, there are 2 dimensions worth of directions to approach the holes in the domain, so only a very special function would be extendable. An extension, if it existed, would still be unique by our previous discussion.

Definition 1. *A rational map of prevarieties, $f : Y \dashrightarrow X$, is an equivalence class of maps $(U, f_U : U \dashrightarrow X)$, U open, f a morphism, where two maps are equivalent if they agree on a non-empty open subset of the intersection of their domains.*

Remark 1.3. *If $X = \mathbb{A}^1 = k$, then the collection of rational maps $f : Y \dashrightarrow \mathbb{A}^1$ is $k(Y)$.*

Definition 2. *A rational map $f : Y \dashrightarrow X$ is dominant if there exists a representative for its class with image dense in X.*

Example 1. *Consider $f : \mathbb{A}^2 \dashrightarrow \mathbb{A}^1$ defined by $f(x, y) = y/x$. It is, of course, only defined on $\mathbb{A}^2_x$. The map f is surjective, hence dominant.*

Theorem 1.4. *If X and Y are varieties, there exists a* 1-1 *correspondence between dominant rational maps, $\phi : Y \dashrightarrow X$ and k-algebra morphisms of function fields $\phi^* : A(X) \to A(Y)$.*

Proof. Suppose $\phi : Y \dashrightarrow X$ is dominant, where ϕ is defined on $U \subseteq X$ open. Then $\phi^* : k(X) \to k(Y)$ can be defined as $\phi^*(\langle V, \eta\rangle) = \langle \phi^{-1}(V), \eta \circ \phi\rangle$. Note that $\phi^{-1}(V)$ is not empty because the image of ϕ is dense. It is left as an exercise to see that ϕ is well-defined and a k-algebra morphism.

Conversely, let $\theta : k(X) \to k(Y)$ be a k-algebra morphism of fields. As it cannot be trivial, it must be injective. Take $\emptyset \neq U \subset X$ open affine, where we think of $U \subseteq \mathbb{A}^n$. Then $O_X(U) = A(U)$, and the coordinate functions of $\mathbb{A}^n$, $x^1, \ldots, x^n \in O_X(U)$, are a finite set of generators for $O_X(U)$ as a k-algebra. Then $\theta(x^1), \ldots, \theta(x^n) \in k(Y)$. There exists an open dense subset $V \subset Y$, which we may assume to be affine, on which they are all defined. Now we can think of $\theta : A(U) \to A(V)$. From our previous work, we know that this is equivalent to a morphism $f : V \to U$ such that $f^* = \theta$. Then for $y \in V$, $f(y) = (\theta(x^1)(y), \ldots, \theta(x^n)(y))$. Then $f : Y \dashrightarrow X$ is a rational map. Suppose the image of f has closure $Z \subsetneq X$. Then $0 \subsetneq I(Z)$ is in the kernel of θ, contradicting its injectivity. It's easy to check that these two processes are inverses, so we have established a bijection. □

Definition 3. *A dominant rational map $f : Y \dashrightarrow X$ is birational if and only if there exists a dominant map $g : X \dashrightarrow Y$ such that $f \circ g = id_X$ and $g \circ f = id_Y$ as rational maps. In this case we say that X and Y are birationally isomorphic.*

Proposition 1.5. *Let X and Y be varieties. Then the following are equivalent:*

(1) X and Y are birationally isomorphic,

(2) $k(X) \simeq k(Y)$ as k-algebras, and

(3) there exists a non-empty open sets $U \subseteq X$ and $V \subseteq Y$ such that U and V are isomorphic as varieties.

We have essentially already proven this theorem.

Definition 4. *If X is a variety, we say that X is rational if it is birational to $\mathbb{P}^n$ (or $\mathbb{A}^n$), or equivalently, if $k(X) \simeq k(x_1, \ldots, x_n)$ for some n.*

Example 2. *Let $\phi : \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ be the Segre embedding, and call the quadric surface, image of ϕ, Q. Then Q is rational but not isomorphic to $\mathbb{P}^2$. Consider the copy of $\mathbb{A}^3$ sitting inside $\mathbb{P}^3$, with fourth homogeneous coordinate 1. Here, the equation for Q is $xy = z$. Then $k(Q) = frac(k[x, y, z]/(xy - z)) = frac(k[x, y]) = k(x, y)$. However, $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$ are not isomorphic. We have previously covered Q with disjoint rulings by lines. Suppose C and D are two curves in $\mathbb{P}^2$. Then $\mathbb{P}^2 \setminus C$ is affine, which we can see using the Veronese map from the homework. But D is projective and hence cannot sit in an affine subset, so D is not contained in the compliment of C, so they intersect.*