MONDAY, FEBRUARY 28, 2005-NOTES TAKEN BY ROHIT GHOSH

Historical Intermezzo:
The Italian School consisting of Cremona,Severi,Enriques,Segre and others dealt with $X \subset \mathbb{P}^n$. They defined dimension and degree in the following manner.

Definition 0.1. $dimX = d$, if $H_1, \ldots, H_n$ are general hyperplanes in $\mathbb{P}^n$ then $\#(X \bigcap H_1 \cdots \bigcap H_n) < \infty$.

Definition 0.2. $degX$ is defined as $\#(X \bigcap H_1 \cdots \bigcap H_n)$.

However, these definitions only work for projective varieties.

Definition 0.3. Let X be a prevariety over k then $dimX = tr.deg_k k(X)$. Recall that $tr.deg_k k(X) = n$ iff $\exists f_1, \ldots, f_n \in k(X)$ which are algebrically independent over k and $k(f_1, \ldots, f_n) \hookrightarrow k(x_1, \ldots x_n)$ is an algebraic extension.

Remark 0.4. 1. Loosely speaking it the $dimX$ is the maximum number of linear coordinates on X.
2. $\emptyset \neq U \subseteq X$, $k(U) = k(X) \Rightarrow dimX = dimU$.
3. $dim\mathbb{A}^n = n$
4. dim X = 0 iff $k(X) = k$ iff X is a point.

Proposition 0.5. *If X is a prevariety , $Y \subsetneq X$ is a proper subvariety $\Rightarrow dimY < dimX$*

Proof. $Y \subsetneq X$. Pick $U \subseteq X$ affine and $U \bigcap Y \neq \emptyset$.
$dimX = dimU$
$dimY = dim(Y \bigcap U)$
So we restrict ourselves to the affine case . Let $R = \mathcal{O}_X(U)$. Then $Y \bigcap U$ is an affine subvariety given by some ideal $\mathfrak{F} = \mathcal{I}(Y \bigcap U) \subseteq R$ and $\mathfrak{F}$ is prime.

$$dim(Y \bigcap U) = tr.deg_k Q(R/\mathfrak{F}) = tr.deg_k(R/\mathfrak{F})$$

We have to prove $tr.deg_k(R/\mathfrak{F}) < tr.deg_k(R)$

Lemma 0.6. *Let R be an integral domain, $(0) \neq \mathfrak{F} \subseteq R$ prime . Then if $tr.deg_k(R) < \infty$, then $tr.deg_k(R/\mathfrak{F}) < tr.deg_k(R)$*

Proof. Suppose $tr.deg_k(R) = n$. Suppose $\exists x_1, \ldots, x_n \in R$ such that $\overline{x_1} = x_1 \bmod \mathfrak{F}, \ldots, \overline{x_n} = x_n \bmod \mathfrak{F}$.
Fix $q \in \mathfrak{F} - 0$.
$\therefore q, x_1, \ldots, x_n \in R \Rightarrow \exists f \in k[y_0, \ldots, y_n]$ such that $f(q, x_1, \ldots, x_n) = 0$.
We may also assume f is irreducible . Also f cannot be y_0.
Hence $F(y_1, \ldots, y_n) = f(0, y_1, \ldots, y_n) \neq 0$.

$F(\overline{x_1}, \dots, \overline{x_n}) = f(\bar{q}, \bar{x_1}, \dots, \bar{x_n}) = f(q, x_1, \dots, x_n) = 0.$
Hence $\overline{x_1}, \dots, \overline{x_n}$ are not algebrically independent. A contradiction.

Definition 0.7. $Y \subseteq X$ be a subvariety, then $codim(Y, X) = dimX - dimY \geq 0$

Theorem 0.8. *Let X be a variety and $U \subseteq X$ be an open set . Let $g \in \mathcal{O}_X(U)$ and Z an irreducible component of $\mathcal{Z}(g) = \{x \in U : g(x) = 0\}$. Then $dimZ = dimX - 1$.*

Example 0.9. $Q : xz = yw$ in $\mathbb{P}^3$
$f = \frac{x^2+z^2-3xw}{y^2+yx+3z^2}$
Any component of the locus $Q \bigcap \mathcal{Z}(f)$ has dimension 1.

Proof. Let $U_0 \subseteq U$ be affine such that $U_0 \bigcap \mathcal{Z}(g) \neq \emptyset$.
$R = \mathcal{O}_X(U_0)$
$f := g|_{U_0}$ corresponds to the ideal (f) .
The irreducible components of $U_0 \bigcap \mathcal{Z}(g)$ correspond to minimal primes $\mathfrak{F} \subseteq R$ such that $(f) \subseteq \mathfrak{F}$. We want $tr.deg_k(R/\mathfrak{F}) \leq tr.deg_k(R) - 1$

Theorem 0.10. (**KrullHaupidealsatz**) *Let R be a finitely generated integral domain over k, let $0 \neq f \in \mathfrak{F}$ be a minimal prime ideal. Then $tr.deg_k(R/\mathfrak{F}) \leq tr.deg_k(R) - 1$*

Corollary 0.11. *If X is a variety and $Z \subseteq X$ is a maximal closed proper subset . Then*
$dimZ = dimX - 1$

Proof. Suppose $dimZ \leq dimX - 2$. We can assume X is affine .
$R = \mathcal{O}_X(X)$ and $(0) \neq I(Z) \subseteq R$ a prime ideal.
Take $f \in \mathcal{I}(Z) \Rightarrow Z \subsetneq \mathcal{Z}(f) \subset X$
$Z \neq Z(f)$ because each component of $\mathcal{Z}(f)$ has codimension 1.
This contradicts the maximality of Z.

Corollary 0.12. *$dimX = max\{r : \exists \emptyset \neq Z_0 \subset Z_1 \subset \dots Z_r = X\}$. The Z_i's are closed irreducible subsets*

Proof. Induction on r.

A third approach to dimension

Let R be a localring . Let $\underline{\underline{m}} \subseteq R$ be a maximal ideal.
$dim(R) = max\{n, \exists (0) = \mathfrak{F}_0 \subset \mathfrak{F}_1 \dots \subset \mathfrak{F}_n = \underline{\underline{m}}\}$. The chain is a chain of prime ideals. This is the Krull dimension of R.

Proposition 0.13. *If X is a variety and $p \in X \Rightarrow dimX = dim\mathcal{O}_{X,p}$*

Proof. Let X be affine and $p \in X$ and $R = \mathcal{O}_X(X)$
$\mathcal{O}_{X,p} = R_{\underline{\underline{m}}_p}$ and $\underline{\underline{m}} = m_p$
There is a one to one correspondence between prime ideals in $R_{\underline{\underline{m}}}$ and prime ideals of R contained in $\underline{\underline{m}}$.

Chain of prime ideals contained in $\underline{\underline{m}}_p \leftrightarrow$ chain of irreducible closed subsets starting with $Z_0 = \{p\}$.
$\therefore dim\mathcal{O}_{X,p} = dimX$.