February 23, 2005-Algebraic Geometry- Notes taken by Ricardo Conceicao

Proposition 0.1. *Let* $X \subset \mathbb{P}^n$ *be a projective variety. Let* $f_0, ..., f_s \in k[X_0, ..., X_n]$ *homogeneous of the same degree such that* $f_0, ..., f_s$ *do not vanish simultaneously on* X*. Then* $f : X \longrightarrow \mathbb{P}^s$ *defined by* $f(x) = [f_0(x) : ... : f_s(x)]$ *is an algebraic morphism.*

Example 0.2. The following example shows the non-validity of the reciprocal: not every morphism $X \longrightarrow \mathbb{P}^n$ can be given uniformly by polynomials. Let $f : \mathbb{P}^1 \longrightarrow \mathbb{P}^2$ be defined by $f[u : v] = [u^2 : uv : v^2]$. Set $X = \operatorname{Im} f$. Then $I(X) = (X_1^2 - X_0X_2)$ and if $P = [x_0 : x_1 : x_2] \in X$ then $x_1 \neq 0$ or $x_2 \neq 0$ or $P = [1 : 0 : 0]$

1. Products of prevarieties

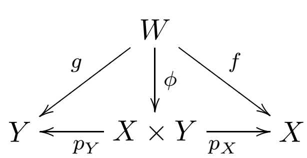
Let X and Y be prevarieties. We want the product $X \times Y$ to be a prevariety too. Unfortunately, we cannot just put the product topology in it as the following examplse shows.

Example 1.1. $\mathbb{A}^1 \times \mathbb{A}^1$, as a prevariety should be isomorphic to $\mathbb{A}^2$. It's not hard to see that the Zariski topology in $\mathbb{A}^2$ is different from the product topology in $\mathbb{A}^1 \times \mathbb{A}^1$

The key fact in defining the product of two varieties is that the product can be defined categorically. First let's see how to define the product in the category of sets.

Let's consider two sets X and Y. Let $p_X : X \times Y \longrightarrow X$ and $p_Y : X \times Y \longrightarrow Y$ be the canonical projections. Then $X \times Y$ satisfies:

For any set W with maps $W \xrightarrow{f} X$ and $W \xrightarrow{g} Y$ there's a unique map $\phi : W \longrightarrow X \times Y$ that makes



into a commutative diagram, i.e., you can factor f and g through the map ϕ, $f = p_X \circ \phi$ and $g = p_Y \circ \phi$

Remark 1.2.
(1) Check that in the category of topological spaces, the product topology satisfies this universal property.
(2) If a product exists, it's unique up to isomorphism.

Now let's see how this universal property will give us a clue on how to define the product of two affine prevarieties.

Example 1.3. Consider prevarieties $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$. Let's suppose that $X \times Y \subseteq \mathbb{A}^{m_n}$ is defined. Suppose $I(X) = (f_1, ..., f_s) \subset k[X_1, ..., X_n]$

and $I(Y) = (g_1, ..., g_t) \subset k[Y_1, ..., Y_m]$. Working out a few examples it's not hard to guess that $I(X \times Y) = (f_i, g_j) \subset k[X_1, ..., X_n, Y_1, ..., Y_m]$.

The canonical projections $p_X : X \times Y \longrightarrow X$ and $p_Y : X \times Y \longrightarrow Y$ are morphisms that induce k-algebra homomorphisms

$$\begin{array}{ccc} A(X) & \longrightarrow & A(X \times Y) \\ & & \uparrow \\ & & A(Y) \end{array}$$

The commutative diagram in the category of affine varieties induces the following commutative diagram over the category of k-algebras:

$$\begin{array}{ccccc} A(X) & \xrightarrow{p_X^*} & A(X \times Y) & \xleftarrow{p_X^*} & A(Y) \\ & {\scriptstyle f^*}\searrow & \downarrow{\scriptstyle \phi^*} & \swarrow{\scriptstyle g^*} & \\ & & W & & \end{array}$$

Where $\phi : W \longrightarrow X \times Y$ is the unique morphism obtained from the universal property of products. This last diagram is the diagram that defines Tensor products as Universal objects, so $A(X \times Y) = A(X) \otimes_k A(Y)$. Note that $A(X) \otimes_k A(Y)$ is a k-algebra and an integral domain, since $A(Y)$ and $A(X)$ are also integral domains. So the affine variety with ring of functions $A(X) \otimes_k A(Y)$ is the product $X \times Y$ of the affine varieties X and Y. Notice that the canonical maps $A(X) \longrightarrow A(X) \otimes_k A(Y)$ and $A(Y) \longrightarrow A(X) \otimes_k A(Y)$ corresponds to the canonical projections on X and Y respectively. Also it's not hard to see that

$$\begin{aligned} A(X \times Y) &= A(X) \otimes_k A(Y) = \frac{k[X_1, ..., X_n]}{(f_1, ..., f_s)} \otimes_k \frac{k[Y_1, ..., Y_m]}{(g_1, ..., g_r)} \\ &\simeq \frac{k[X_1, ..., X_n, Y_1, ..., Y_m]}{(f_i, g_j)} \end{aligned}$$

Example 1.4. Let $X \subseteq \mathbb{A}^3$ be defined by $XYZ = X^2 + 3$ and $Y \subset \mathbb{A}^2 : st = 7$ then $X \times Y$ is the subset $\{(x, y, z, s, t) \in \mathbb{A}^5 | xyz = x^2 + 3, st = 7\}$

2. Topology on $X \times Y$

Suppose X and Y affine varieties. The product of them as a set is just the set $X \times Y$. Let's see how to define the topology to get an affine variety. Let's start by defining the distinguished sets. For that pick $f \in A(X) \otimes_k A(Y)$. Recall that $f = \sum_i f_i(\underline{x}) \otimes g_i(\underline{y})$ and using the isomorphism

$$\frac{k[\underline{X}]}{I(X)} \otimes_k \frac{k[\underline{Y}]}{I(Y)} \longrightarrow \frac{k[\underline{X}, \underline{Y}]}{(I(X), I(Y))} \qquad f \otimes g \longmapsto f(\underline{x})g(\underline{y})$$

We see that

$$(X \times Y)_f = \{(x, y) \in X \times Y | \sum f_i(x)g_i(y) \neq 0\}$$

forms a base for the topology on $X \times Y$.

3. Field of functions of $X \times Y$

We know that $k(X \times Y)$ is by definition $\mathrm{quot}(\mathrm{A(X)} \otimes_{\mathrm{k}} \mathrm{A(Y)})$ which is equal to $k(X) \otimes_k k(Y)$.

4. Stalks

In this case, the stalk at a point $(x, y) \in X \times Y$ is given by

$$\mathcal{O}_{X\times Y,(x,y)} = (A(X) \otimes_k A(Y))_{m_{(x,y)}}$$

Recall that in general $\mathcal{O}_{X\times Y,(x,y)}$ = ring of germs of functions $X \times Y$ localized at the ideal of those functions vanishing at (x, y). Let's prove that $\mathcal{O}_{X\times Y,(x,y)} \simeq (\mathcal{O}_{X,x} \otimes_k \mathcal{O}_{Y,y})_{m_x\otimes\mathcal{O}_{Y,y}+\mathcal{O}_{X,x}\otimes m_y}$.

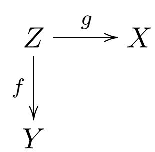
Indeed, let $J = m_x \otimes \mathcal{O}_{Y,y} + \mathcal{O}_{X,x} \otimes m_y$, i.e.,

$$J = \left\{ \sum_i (f_i \otimes g_i + u_i \otimes v_i) \middle| f_i(x) = g_i(y) = 0, g_i, v_i \in \mathcal{O}_{Y,y}, f_i, u_i \in \mathcal{O}_{X,x} \right\}$$

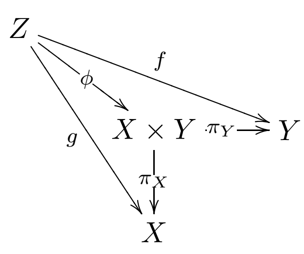
Notice that for $s = \sum_i f_i \otimes g_i \in \mathcal{O}_{X,x} \otimes_k \mathcal{O}_{Y,y}$ with$\sum_i f_i(x)g_i(y) = 0$, we'll have $s = \sum(f_i - a_i) \otimes g_i + \sum a_i \otimes (g_i - b_i) \in (m_x \otimes \mathcal{O}_{Y,y} + \mathcal{O}_{X,x} \otimes m_y)$, where $b_i = g_i(y)$ and $a_i = f_i(x)$ are elements of k. Therefore J is the ideal of germs on $X \times Y$ vanishing at (x, y).

Now we are ready to discuss the general case:

Theorem 4.1. *If X and Y are prevarieties defined over k then their product $X \times Y$ exists (as a prevariety satisfying the universal property: Given*



There exists a unique morphism $\phi = (f, g) : Z \longrightarrow X \times Y$ making the following diagram comutative



Proof: As a set $X \times Y = \{(x, y) : x \in X, y \in Y\}$. We define the *topology* in the following way: Let $\mathcal{U} \subseteq X$ and $\mathcal{V} \subseteq Y$ be affine open subsets. Let $f = \sum_i f_i \otimes g_i$, $f_i \in A(\mathcal{U}) = \mathcal{O}_X(\mathcal{U})$, $g_i \in A(\mathcal{V}) = \mathcal{O}_Y(\mathcal{V})$. A basis for the topology in $X \times Y$ is given by the sets

$$\left\{ (x, y) \in \mathcal{U} \times \mathcal{V} \middle| \sum_i f_i(x)g_i(y) \neq 0 \right\}$$

over all open affines $\mathcal{U} \subseteq X$, $\mathcal{V} \subseteq Y$, $f_i \in \mathcal{O}_X(\mathcal{U})$, $g_i \in \mathcal{O}_Y(\mathcal{V})$.

Define:

the function field by

$$k(X \times Y) := k(X) \otimes_k k(Y)$$

the stalks by

$$\mathcal{O}_{X\times Y,(x,y)} := (\mathcal{O}_{X,x} \otimes_k \mathcal{O}_{Y,y})_{m_x \otimes \mathcal{O}_{Y,y} + \mathcal{O}_{X,x} \otimes m_y}$$

and *the regular functions* on an open set $\mathcal{U} \subseteq X \times Y$ by

$$\mathcal{O}_{X\times Y}(\mathcal{U}) := \bigcap_{(x,y)\in\mathcal{U}} \mathcal{O}_{X\times Y,(x,y)}$$

Using this definitions it's not hard to show that $(X \times Y, \mathcal{O}_{X\times Y})$ is a prevariety. Now we're left to check that $X \times Y$ satisfies the universal property of a product. To check that let us consider for any prevariety Z and morphisms $s : Z \longrightarrow Y$, $r : Z \longrightarrow X$ the following diagram:

$$\begin{array}{ccc} Z & \xrightarrow{\ r\ } & X \\ {\scriptstyle s}\downarrow & & \uparrow{\scriptstyle p_X} \\ Y & \xleftarrow[\ p_Y\ ]{} & X \times Y \end{array}$$

We want to construct a morphism $\phi : Z \longrightarrow X \times Y$ which makes the diagram commutes.

Let $\phi = (r, s) : Z \longrightarrow X \times Y$ be the map defined by $z \longmapsto \phi(z) = (r(z), s(z))$. If $\mathcal{U} \subseteq X$ and $\mathcal{V} \subseteq Y$ are non-empty affine open sets, then define $Z_{\mathcal{U},\mathcal{V}} = r^{-1}(\mathcal{U}) \cap s^{-1}(\mathcal{V})$ which is an open set in Z. It suffices to check that $\phi|_{Z_{\mathcal{U},\mathcal{V}}}$ is a morphism. To show that $Z_{\mathcal{U},\mathcal{V}} \xrightarrow{\phi} \mathcal{U} \times \mathcal{V}$ is a morphism is equivalent to show that $\phi^* : A(\mathcal{U} \times \mathcal{V}) \longrightarrow \mathcal{O}_Z(Z_{\mathcal{U},\mathcal{V}})$ is a homomorphism. Recall that we can identify $A(\mathcal{U} \times \mathcal{V})$ with $A(\mathcal{U}) \otimes_k A(\mathcal{V})$, since $\mathcal{U}$ and $\mathcal{V}$ are affine prevarieties. So take $f(x) \otimes g(y) \in A(\mathcal{U} \times \mathcal{V})$. Then it's not hard to see that $\phi^*(f \otimes g) = f^*(f)s^*(g)$ and that this is a homomorphism. Also, since r and s are morphism, we have that $r^*(f)$ and $s^*(g)$ are regular functions, as well as their product. So ϕ is a morphism and clearly satisfy the universal property. □

Example 4.2. *Segre Embedding*

Let's make $\mathbb{P}^n \times \mathbb{P}^m$ into a projective prevariety. Consider $\phi : \mathbb{P}^n \times \mathbb{P}^m \longrightarrow \mathbb{P}^{mn+m+n}$ the map defined by

$$\phi([x_0 : \ldots : x_n], [y_0 : \ldots : y_m]) = [z_{ij} = x_i y_j]_{\substack{0\le i\le n \\ 0\le j\le m}}$$

We should notice that this map can be represented as the matrix:

$$A = \begin{bmatrix} z_{00} & \cdots & z_{0m} \\ \vdots & \ddots & \vdots \\ z_{n0} & \cdots & z_{nm} \end{bmatrix}$$

ϕ is well-defined since it clearly does not depend on the homogeneous coordinate and you always can find i and j such that $x_i y_j \neq 0$. Also ϕ is injective as one can easily verify.

Let $Z = \mathrm{Im}(\phi)$. Then Z is a closed algebraic set in $\mathbb{P}^{mn+m+n}$. In fact, $Z = \{A = [z_{ij}]_{\substack{0\leq i\leq n \\ 0\leq j\leq m}} \,|\, \mathrm{rank}(A) = 1\}$, that is, $I(Z)$ is generated by the 2×2 minors $z_{ij}z_{lk} - z_{lj}z_{ik}$. One can check that ϕ gives an embedding $\phi : \mathbb{P}^n \times \mathbb{P}^m \overset{\phi}{\hookrightarrow} \mathbb{P}^{mn+m+n}$ (see Munford's, *The Red Book of Varieties and Schemes*, pg 37).