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(Notes taken by Laura Hitt)

Last time we defined prevarieties. Now we will see a few examples:

**Example 0.1.**  $\mathbb{P}^1$  – the projective line.

Let  $U = \mathbb{A}^1$ ,  $U_0 = \mathbb{A}^1 - \{0\}$ ,  $V = \mathbb{A}^1$ ,  $V_0 = \mathbb{A}^1 - \{0\}$ . We glue  $U$  and  $V$  together along  $U_0$  and  $V_0$  as follows: first we define an isomorphism

$$f : U_0 \rightarrow V_0, f(u) = 1/u$$

$$f^* : \mathcal{O}_{V_0}(V_0) \xrightarrow{\sim} \mathcal{O}_{U_0}(U_0)$$

$$k[v, 1/v] \xrightarrow{\sim} k[u, 1/u]$$

$$v \mapsto 1/u, 1/v \mapsto u$$

So  $\mathbb{P}^1 = U \sqcup V/u \sim 1/u$  is the identification. Set-theoretically we have that  $\mathbb{P}^1 = U \cup \{\infty\}$ .

**Example 0.2.** We now explain a more general procedure of gluing two prevarieties along open subsets: Assume that  $U, V$  are prevarieties,  $U_0 \subseteq U$  and  $V_0 \subseteq V$  are open subsets and

$$f : (U_0, \mathcal{O}_{U_0}) \xrightarrow{\sim} (V_0, \mathcal{O}_{V_0}) \text{ is an isomorphism.}$$

Glue  $U$  and  $V$  along  $U_0$  and  $V_0$  via  $f$ .

$$X = U \sqcup V/U_0 \ni u \sim f(u) \in V_0$$

$$i : U \hookrightarrow X, i(u) = \bar{u} \text{ and } j : V \hookrightarrow X, j(v) = \bar{v} \text{ are inclusions.}$$

So  $X = U \cup V$ .

**Define a topology on  $X$  :**

$$W \subseteq X \text{ open} \Leftrightarrow$$

$$i^{-1}(W) = W \cap U \text{ open in } U, \text{ and } j^{-1}(W) = W \cap V \text{ open in } V.$$

**Functions on  $X$  :** "pairs of functions on  $U$  and  $V$  that agree on the overlap"

$$\mathcal{O}_X(W) = \{(\phi, \eta) \in \mathcal{O}_U(i^{-1}(W)) \times \mathcal{O}_V(j^{-1}(W)) : f^*(\eta) = \phi\}$$

$$f^* : \mathcal{O}_{V_0}(V_0 \cap j^{-1}(W)) \rightarrow \mathcal{O}_{U_0}(U_0 \cap i^{-1}(W))$$

$$\begin{array}{ccc} U_0 \cap i^{-1}(W) & \xhookrightarrow{i} & W \subset X \\ & \searrow f & \swarrow j \\ & & V_0 \cap j^{-1}(W) \end{array}$$

commutes.

In this way,  $(X, \mathcal{O}_X)$  becomes a prevariety.

**Example 0.3.** Determine  $\Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1})$ , the global sections of  $\mathbf{P}^1$ .

$$\begin{array}{ccc} U_0 & \xhookrightarrow{i} & \mathbb{P}^1 \\ f \downarrow & \nearrow & \\ V_0 & & \end{array}$$

$$\begin{aligned} & \phi \in \mathcal{O}_U(U), \eta \in \mathcal{O}_V(V) \text{ such that } f^*(\eta) = \phi. \\ & \mathcal{O}_U(U) = \Gamma(\mathbb{A}^1, \mathcal{O}_{\mathbb{A}^1}) = k[u], \text{ and } \mathcal{O}_V(V) = k[v]. \\ \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) &= \{(\phi, \eta) \in k[u] \times k[v] : f^*(\eta) = \phi\} \\ &= \{(\phi, \eta) \in k[u] \times k[v] : \eta(1/u) = \phi(u)\} \end{aligned}$$

$$\begin{aligned} \text{But } \eta(1/u) = \phi(u) &\implies \phi = \eta \in k \text{ (degree 0)} \\ &\implies \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = k \end{aligned}$$

(So global functions on  $\mathbb{P}^1$  are constants.)

**Remark 0.4.**  $\mathbb{P}^1$  is not affine.

To see this we look at the inclusion  $i : \{0\} \hookrightarrow \mathbb{P}^1$ . If  $\mathbb{P}^1$  was affine, then since the map induced at the level of affine coordinate rings  $i^* : A(\mathbb{P}^1) \xrightarrow{\sim} A(\{0\})$  is an isomorphism, it would follow that  $i : \{0\} \hookrightarrow \mathbb{P}^1$  is an isomorphism as well, which is clearly not the case.

**Example 0.5.** A pathological example.

$$\begin{aligned} U &= \mathbb{A}^1, U_0 = \mathbb{A}^1 - \{0\}, V = \mathbb{A}^1, V_0 = \mathbb{A}^1 - \{0\} \\ Y &= U \sqcup V / \sim, \text{ where } u \in U_0 \sim u \in V_0 \\ g : U_0 &\rightarrow V_0, g(u) = u \end{aligned}$$

This is "very" non-Hausdorff, because a sequence of points going to 0 has two limits at the double point 0.

$(Y, \mathcal{O}_Y)$  is a prevariety, but it will not be a variety because of this bad double point.

**Example 0.6.** The  $n$ -dimensional projective space:

As a set,  $\mathbb{P}^n = k^{n+1} - \{(0, \dots, 0)\} / \sim$ , where  $(a_0, \dots, a_n) \sim (b_0, \dots, b_n) \iff \exists \lambda \in k^*$  such that  $b_i = \lambda a_i$ . We denote points in projective space by  $p = [a_0, \dots, a_n] \in \mathbb{P}^n$  (homogeneous coordinates of  $p$ ). We want to turn  $\mathbb{P}^n$  into a prevariety:

Let  $U_i = \{p = [a_0, \dots, a_n] \in \mathbb{P}^n : a_i \neq 0\}$ , hence  $U_0 \cup \dots \cup U_n = \mathbb{P}^n$ . Moreover, for each  $0 \leq i \leq n$  we have a map  $\phi_i : \mathbb{A}^n \rightarrow U_i$  where  $\phi_i(x_1, \dots, x_n) = [x_1, \dots, x_{i-1}, 1, x_i, \dots, x_n]$ . This is a bijection, which we will use to glue  $U_0, U_1, \dots, U_n$ .

$$\begin{array}{ccc} \mathbb{A}^n \supset \phi_0^{-1}(U_0 \cap U_1) & \xrightarrow{\phi_0} & U_0 \cap U_1 \\ \downarrow \Phi \sim & \nearrow \phi_1 & \\ \mathbb{A}^n \supset \phi_1^{-1}(U_0 \cap U_1) & & \end{array}$$

$$\phi_0(a_1, \dots, a_n) = [1, a_1, a_2, \dots, a_n]$$

$$\phi_1(b_1, \dots, b_n) = [b_1, 1, b_2, \dots, b_n]$$

$$\text{So } \phi_0^{-1}(U_0 \cap U_1) = \{a_1 \neq 0\} \text{ and } \phi_1^{-1}(U_0 \cap U_1) = \{b_1 \neq 0\}$$

We will define  $\Phi$  and show it is an algebraic map.

We want  $(a_1, \dots, a_n) \mapsto (b_1, \dots, b_n)$  such that  $[1, a_1, a_2, \dots, a_n] = [b_1, 1, b_2, \dots, b_n]$

i.e. such that  $\exists \lambda \in k^*$  such that  $b_1 = \lambda$ ,  $1 = \lambda a_1$ ,  $b_2 = \lambda a_2$ ,  $\dots$ ,  $b_n = \lambda a_n$

$$\Rightarrow \lambda = 1/a_1$$

$$\Rightarrow b_1 = 1/a_1, b_2 = a_2/a_1, \dots, b_n = a_n/a_1.$$

$$\text{So } \Phi(a_1, \dots, a_n) = (1/a_1, a_2/a_1, \dots, a_n/a_1)$$

Glue  $U_0$  and  $U_1$  via  $\Phi$ . Then we need to show that we can glue  $U_0, U_1$  with  $U_2$  in any order and get the same, etc. This shows that  $\mathbb{P}^n$  is a prevariety.

**Example 0.7.**  $\mathbb{P}^4$

$$[1 : 2 : 0 : 3] \in U_0 \cap U_1 \cap (U_3 - U_2)$$

We can write

$$[1 : 2 : 0 : 3] = [1/2 : 1 : 0 : 3/2] = [1/3 : 2/3 : 0 : 1]$$

$$\in \text{Im}(\phi_0) \quad \in \text{Im}(\phi_1) \quad \in \text{Im}(\phi_3)$$

Exercise: Show  $\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}) = k$

(essentially the same proof as for  $\mathbb{P}^1$ , with more variables.)

**Function field of a prevariety :**

Let  $X$  be a prevariety. We define an equivalence relation on the set of pairs  $(U, f)$ , where  $\emptyset \neq U \subseteq X$  open, and  $f \in \mathcal{O}_X(U)$  by:

$$(U, f) \sim (V, g) \iff \exists \emptyset \neq W \subseteq V \cap U, W \text{ open, such that}$$

$$\text{res}_{v,w}(f) = \text{res}_{v,w}(g) \text{ i.e. } f|_w = g|_w$$

(That is, two functions coincide on small open set.)

Note: Since  $X$  is irreducible, any two non-empty open subsets  $U, V \subseteq X$  intersect, so this definition makes sense. (We can also check that this is indeed an equivalence relation.)

$k(X) := \{ \langle U, f \rangle \}$  – the set of equivalence classes.

Note:

1. This is very similar to the definition of a stalk
2. Think of analogy with meromorphic functions on  $\mathbb{C}$ .