Algebraic Geometry- January 31, 2005- Notes taken by Pippa Charters

Let $X \subseteq \mathbb{A}^n$ be an irreducible algebraic set. Our goal is to think about this set X without reference to the ambient space $\mathbb{A}^n$. To this end, recall the domain $A(X)$, the affine coordinate ring of X. Then $Q(A(X)) = k(X) =$ the field of rational functions on X. That is,

$$Q(A(X)) = \left\{ \frac{f}{g} \mid f, g \text{ polynomials on } X \right\}$$

Definition 0.1. Let $p = (a_1, \dots, a_n) \in X$, and define

$$\mathcal{O}_{X,p} = A(X)_{\mathcal{M}_p}, \quad \mathcal{M}_p = \{f \in A(X) \ : \ f(p) = 0\}.$$

This is called a *local ring at p*. That is, $\mathcal{O}_{X,p}$ is a local ring with maximal ideal

$$\left\{ \frac{f}{g} \ : \ f, g \in A(X), g(p) \neq 0 \right\}$$

We would like to create a sheaf $\mathcal{O}_X$. To this end, let $U \subseteq X$ be an open set in X. Then we define a map

$$U \mapsto \mathcal{O}_X(U) = \bigcap_{p \in U} \mathcal{O}_{X,p} \subseteq k(X).$$

Note that this is well-defined. Thus we have

$$\mathcal{O}_X(U) = \left\{ s \in k(X) \ : \ \forall \ p \in U, \exists \ f, g \in A(X) \ : \ g(x) \neq 0, s = \frac{f}{g} \right\}.$$

In particular, think of elements in $\mathcal{O}_X(U)$ as set-theoretic functions defined on U:

$$s : U \to k \ : \ s(x) = \frac{f(x)}{g(x)}.$$

That is, functions on U expressable locally as polynomials.

Remark 0.2. In general, for $s \in \mathcal{O}_X(U)$, it's not true that $s = \frac{f}{g}$ globally on U, $f, g \in A(X)$, as we might have $g(x) = 0$ for some values of x.

Example 0.3. Let $X \subseteq \mathbb{A}^4$ be the space $X : xw = yz$. Define

$$X_W = \{p \in X \ : \ w \neq 0\}, \quad X_Y = \{p \in X \ : \ y \neq 0\}.$$

Let

$$U = X_W \cup X_Y.$$

Then we can define a regular function s on U by:

$$s(p) = \begin{cases} \frac{x}{y} \text{ on } X_Y \\ \frac{z}{w} \text{ on } X_W \end{cases}$$

Then $s(p) \in \mathcal{O}_X(U)$, s is algebraic on each subset, and since $\frac{x}{y} = \frac{z}{w}$ on $X_W \cap X_Y$ it agrees on the overlap. But we cannot find a single function that will work on the entire U.

Remark 0.4. A rigorous proof of this requires the concept of dimension, which we haven't yet discussed. Thus all that we can currently say is that there is no *intuitive* definition of such a function.

It turns out that the association

$$U \mapsto \mathcal{O}_X(U)$$

is a sheaf. If $U \subseteq V$ are both open subsets of X, then we define the restriction map by inclusion:

$$\mathcal{O}_X(U) \hookrightarrow \mathcal{O}_X(V)$$

as both $\mathcal{O}_X(U)$ and $\mathcal{O}_X(V)$ are contained in $k(X)$.

Question 0.5. What is $\mathcal{O}_X(\emptyset)$?

Our two texts have definition discrepancies. In Hartshorne, if $\mathcal{F}$ is a sheaf of rings, then $\mathcal{F}(\emptyset) = 0$ by part of the definition of a sheaf. But it is clear that we actually have $\mathcal{O}_X(\emptyset) = k(X)$.

We can understand $\mathcal{O}_X(U)$ for some distinguished open sets. We have a distinguished basis for the Zariski topology on X given by the following. Let $f \in A(X)$. Then define

$$X_f = X - Z(f) = \{p \in X \; : \; f(p) \neq 0\}.$$

The set $\{X_f\}_{f \in A(X)}$ is a basis for the topology of X.

Proposition 0.6. $\mathcal{O}_X(X_f) = A(X)_f = \left\{ \frac{g}{f^r} \; : \; r \geq 0 \right\}$

Remark 0.7. To relate this with our previous notation for a localized ring, let $S = \{f^r \; : \; r \geq 0\}$. Then $S^{-1}R = R_f$.

Proof. It is clear that $A(X)_f \subseteq \mathcal{O}_X(X_f)$. Thus let's consider the opposite inclusion.

Suppose that $s \in \mathcal{O}_X(X_f)$. Then define the ideal

$$B := \{h \in A(X) \; : \; hs \in A(X)\}.$$

This is an ideal which depends on s. If we show that $f^r \in B$, then we are done, since

$$\begin{aligned} f^r \in B \quad &\iff \quad f^r s \in A(X) \\ &\iff \quad s = \frac{g(x)}{f^r}, \; g(x) \in A(X) \end{aligned}$$

We know that $s \in \mathcal{O}_X(X_f)$. Pick some $x \in X_p$. Then there exist some functions $g, h \in A(X)$ such that $s = \frac{g}{h}$, $h \neq 0$, by definition. Thus we have

$$\begin{aligned} sh = g \in A(X) \quad &\Longrightarrow \quad h \in B, h(x) \neq 0 \\ &\Longrightarrow \quad x \notin Z(B) \\ &\Longrightarrow \quad X_f \cap Z(B) = \emptyset \\ &\Longrightarrow \quad \text{(By Nullstellensatz)} \sqrt{(f)} = I(Z(f)) \subseteq I(Z(B)) = \sqrt{B} \\ &\Longrightarrow \quad f \in \sqrt{B} \\ &\Longrightarrow \quad \exists\, r \text{ such that } f^r \in B. \end{aligned}$$

□

Remark 0.8. $\mathcal{O}_X(U \cup V) = \mathcal{O}_X(U) \cap \mathcal{O}_X(V) \subseteq k(X)$. Any open $U \subset X = \bigcup_{f_\alpha \in A(X)} X_{f_\alpha}$, and $\mathcal{O}_X(U) = \bigcap_\alpha \mathcal{O}(X_{f_\alpha}) = \bigcap_\alpha A(X)_{f_\alpha}$.

Corollary 0.9. $\mathcal{O}_X(X) = A(X)$. *That is, the only regular functions on all of X are polynomials (global regular functions) So the old way of thinking about morphisms is consistent with the new definition.*

Proposition 0.10. *The stalk of $\mathcal{O}_X$ at $p \in X$ is $\mathcal{O}_{X,p} = A(X)_{\mathcal{M}_p}$.*

Proof. Recall that

$$\mathcal{O}_{X,p} = \{\langle U, s\rangle \ : \ U \subseteq X \text{ an open neighborhood of } p,\ s \in \mathcal{O}_X(U)\}.$$

Define a map $\eta : \mathcal{O}_{X,p} \to A(X)_{\mathcal{M}_p}$ by

$$\langle U, s\rangle \mapsto \frac{f}{g}$$

where by assumption $s = \frac{f}{g}$, $g(p) \neq 0$ and $f, g \in A(X)$.

We will show that η is a bijection. That η is injective is fairly straightforward, so we will concentrate on showing surjectivity. Let $\frac{f}{g} \in A(X)_{\mathcal{M}_p}$. Then $\frac{f}{g}$ defines a regular function in $\mathcal{O}_X(X_g)$. Thus we have

$$\eta(\langle X_g, \frac{f}{g}\rangle) = \frac{f}{g}.$$

So η is surjective. It follows that $\mathcal{O}_{X,p} = A(X)_{\mathcal{M}_p}$. □

Example 0.11. Consider $\mathcal{O}_{\mathbb{A}^2}$. Then

$$k(\mathbb{A}^2) = k(x, y) = \text{ field of rational functions}$$

$$A(\mathbb{A}^2) = k[x, y] = \text{ field of regular functions}$$

Moreover, the stalk of $\mathbb{A}^2$ at 0 is as follows:

$$\mathcal{O}_{\mathbb{A}^2,0} = k[x, y]_{\mathcal{M}_0} = \left\{ \frac{f(x, y)}{g(x, y)} \ : \ g(0, 0) \neq 0 \right\}.$$

Let $U = \mathbb{A}^2$. Then $\Gamma(U, \mathcal{O}_{\mathbb{A}^2}) = \{$ regular functions on all of $\mathbb{A}^2\} = k[x, y]$.

Example 0.12. Let $V = \mathbb{A}^2 \setminus \{x = 0\} = \mathbb{A}^2_x$. Then recall that $X_f = \{p \in X \ : \ f(p) \neq 0\}$. Thus we have

$$\begin{aligned}\Gamma(V, \mathcal{O}_{\mathbb{A}^2}) &= k[x,y]_x = \{\frac{f(x,y)}{x^a} \ : \ a \geq 0\} \\ &= k[x,y][\frac{1}{x}].\end{aligned}$$

Example 0.13. Let $V_1 = \mathbb{A}^2 \setminus \{y^2 = x\}$. Then $\Gamma(V_1, \mathcal{O}_{\mathbb{A}^2}) = k[x, y, \frac{1}{y^2 - x}]$.

Example 0.14. Let $W = \mathbb{A}^2 \setminus \{(0,0\} = (\mathbb{A}^2 \setminus \{x = 0\}) \cup (\mathbb{A}^2 \setminus \{y = 0\}) = \mathbb{A}^2_x \cup \mathbb{A}^2_y$. Then

$$\Gamma(W, \mathcal{O}_{\mathbb{A}^2}) = \Gamma(\mathbb{A}^2_x, \mathcal{O}_{\mathbb{A}^2}) \cap \Gamma(\mathbb{A}^2_y, \mathcal{O}_{\mathbb{A}^2}) = k[x,y]_x \cap k[x,y]_y$$

To figure out what this intersection is, note that this is the set of functions $h(x,y) = \frac{f(x,y)}{x^a} = \frac{g(x,y)}{y^b}$ where $f, g \in A(\mathbb{A}^2)$. But this implies that $y^b f(x,y) = x^a g(x,y)$, hence that $x^a \mid f$, $y^b \mid g$ and hence $h(x,y) \in A(\mathbb{A}^2)$

It follows that

$$\Gamma(\mathbb{A}^2 \setminus \{(0,0)\}, \mathcal{O}_{\mathbb{A}^2}) = \Gamma(\mathbb{A}^2, \mathcal{O}_{\mathbb{A}^2}).$$

This is the analogue of Hartog's theorem in Analysis, which is the same statement except about Holomorphic functions.