Algebraic Geometry

1 2/4/05

(Notes taken by Kevin Klonoff)

Definition 1. *A ringed space is a pair $(X, \mathcal{O}_X)$, where X is a topological space and $\mathcal{O}_X$ is a sheaf of rings on X.*

Definition 2. *An isomorphism of ringed spaces $(X, \mathcal{O}_X) \simeq (\Sigma, \mathcal{O}_\Sigma)$ consists of a homeomorphism $f : X \to \Sigma$ together with ring isomorphisms*

$$f^* : \mathcal{O}_\Sigma(U) \to \mathcal{O}_X(f^{-1}(U))$$

for every open set $U \subset \Sigma$, which commute with restrictions. (i.e induces an isomorphism of sheaves)

Definition 3. *A ringed space $(X, \mathcal{O})$ is an affine variety if there exists an irreducible algebraic set $\Sigma \subset \mathbb{A}^n$ and an isomorphism of ringed spaces $(X, \mathcal{O}_X) \simeq (\Sigma, \mathcal{O}_\Sigma)$, where $\mathcal{O}_\Sigma$ denotes the sheaf of regular functions on Σ.*

If X is an affine variety, it is not true that every open $U \subset X$ is an affine variety.

Example 1. *Let $X = \mathbb{A}^2$, and $U = \mathbb{A}^2 - 0$ Then U is not an affine variety.*

Proof. Suppose that U is affine. Let $i : U \hookrightarrow \mathbb{A}^2$ be the inclusion map. Then i induces $i^* : \mathcal{O}_{\mathbb{A}^2}(\mathbb{A}^2) \to \mathcal{O}_U(U)$. We have the following identifications. $\mathcal{O}_{\mathbb{A}^2}(\mathbb{A}^2)$ is just the ring of polynomials in two variables $k[X, Y]$, and as we saw earlier $\mathcal{O}_U(U) = \mathcal{O}_{\mathbb{A}^2}(U) = k[X, Y]$. So i^* is just the identity map, an isomorphism. This says that $i : U \hookrightarrow \mathbb{A}^2$ is an isomorphism. This is a contradiction since i is not onto. □

Definition 4. *A connected topological space X with a sheaf of k - valued functions on X, $\mathcal{O}_X$, is a pre-variety if there is a finite open cover $\{U_\alpha\}$ of X such that the ringed space $(U_\alpha, \mathcal{O}_{X|U_\alpha} = \mathcal{O}_{U_\alpha})$ is an affine variety.*

Remarks:

- $(X, \mathcal{O}_X)$ a pre-variety. If $U \subset X$ is open, then $(U, \mathcal{O}_{X|U})$ is also a pre - variety.
- If X is a pre-variety, then X is an irreducible space.

Proof. Homework □

Theorem 1.1. *If X is a pre-variety, then X is noetherian (descending chains of closed sets stabilize).*

Proof. Let

$$Z_1 \supset Z_2 \supset \ldots \supset Z_n \supset Z_{n+1} \supset \ldots$$

be a descending chain of closed sets.

There is a finite open cover of X, $\{U_\alpha\}$ such that $(U_\alpha, \mathcal{O}_{U_\alpha})$ is affine. It is sufficient to show that each sequence

$$U_\alpha \cap Z_1 \supset U_\alpha \cap Z_2 \supset, \ldots, \supset U_\alpha \cap Z_n \supset U_\alpha \cap Z_{n+1} \supset, \ldots,$$

stabilizes.

Let us fix now an element U_α of the affine covering of X. Since $U_\alpha \simeq \Sigma$ is affine, we may assume that $U_\alpha \subset \mathbb{A}^n$.

For each i we have that $U_\alpha \cap Z_i$ is closed in U_α hence it corresponds to an ideal $\mathfrak{a}_i \in A(\Sigma)$. Now this gives an increasing sequence of ideals in the noetherian ring $A(\Sigma)$. Thus this sequence stabilizes and so the corresponding sequence of closed sets stabilizes.

□

Corollary 1.2. *Any prevariety X is quasi-compact (compact but not hausdorff).*

Proof. A set is quasi-compact if any open cover has a finite subcover. This is equivalent to the following: If $\{F_i\}$ is any collection of closed sets satisfying $\cap F_i = \emptyset$, then there is a finite collection of the F_i's satisfying $\cap_{k=1}^{N} F_{i_k} = \emptyset$.

To obtain a contradiction suppose that F_i is a collection of closed sets of X satisfying $\cap F_i = \emptyset$ but no finite collection $\{F_{i_k}\}$ satisfies $\cap_{k=1}^{N} F_{i_k} = \emptyset$. Then we can construct a decreasing sequence of closed sets that does not stabilize as follows:

Start with $F_1 \neq \emptyset$. Then there is an F_{i_2} such that $F_1 \cap F_{i_2} \neq \emptyset$ and $F_1 \supsetneq F_1 \cap F_{i_2}$. Similarly there is an F_{i_3} such that $F_1 \cap F_{i_2} \cap F_{i_3} \neq \emptyset$. So $F_1 \supsetneq F_1 \cap F_{i_2} \supsetneq F_1 \cap F_{i_2} \cap F_{i_3}$. Continuing in this way we construct a decreasing sequence of closed sets that does not stabilize. This is a contradiction, so there is a finite subcollection of F_i with empty intersection and X is thus quasi-compact.

□

Enough abstract nonsense.

Example 2. *The projective line $\mathbb{P}^1$.*

The projective line, $\mathbb{P}^1$ is defined to be the quotient of $(k^2 - \{0\})$ by the equivalence relation $\sim$ where $x \sim y$ if $x = \lambda y$ for some $\lambda \in k^*$. So as a set, $\mathbb{P}^1 = (k^2 - (0,0))/\sim$.

We can write $\mathbb{P}^1 = \{[x, y] : (x, y) \neq (0, 0)\}$.

In the familiar case of $k = \mathbb{C}$, we get the complex projective line $\mathbb{P}^1(\mathbb{C})$, which is homeomorphic to the sphere S^2, and can be thought of as the complex line union a point: $\mathbb{C} \cup \infty$.

Return to the general case where k is any (algebraically closes) field.

Let $U = \mathbb{A}^1$ with coordinate u, and set $U_0 = (\mathbb{A}^1 - \{0\})$. Similarly, set $V = \mathbb{A}^1$ with coordinate v, and define $V_0 = (\mathbb{A}^1 - \{0\})$. We want to glue U and V along U_0 and V_0 but we first have to specify the gluing recipe. We will glue via the isomorphism $f : U_0 \to V_0$ where $u \mapsto \frac{1}{u} = v$.

Claim that f is an isomorphism of pre-varieties. First note that the ring of functions on U_0 is $k[u]_u = k[u, \frac{1}{u}]$. Similarly the ring of functions on V_0 is the ring $k[v, \frac{1}{v}]$. The induced map $f^* : \mathcal{O}_{V_0}(V_0) \to \mathcal{O}_{U_0}(U_0)$ is given by $v \mapsto \frac{1}{u}, u \mapsto \frac{1}{v}$. It is clear that f is a k-algebra isomorphism.

We then use f to glue U and V along U_0 and V_0. As a set

$$\mathbb{P}^1 := U \amalg V / (u \in U_0 \sim \frac{1}{u} = v \in V_0)) = U \cup \{\infty\}.$$

Here $\infty \in \mathbb{P}^1$ denotes the equivalence class of $0 \in V$. We want to show that $\mathbb{P}^1$ is a pre-variety.

1. Give $\mathbb{P}^1$ a topology, the quotient topology, that is, a set $W \subset \mathbb{P}^1$ is open if and only if $W \cap U$ in open in U and $W \cap V$ is open in V.
2. We now endow $\mathbb{P}^1$ a sheaf of regular functions. (Next time)