Algebraic Geometry March 2nd 2005 -Notes taken by Pippa Charters

Recall from the previous class the result that if X is an affine variety and f is a non-zero function, then each component of $Z(f)$ has dimension dim $X-1$. That is, $Z(f)$ has pure codimension 1.

Proposition 0.1. *If X is an affine variety and $f_1, \ldots, f_t \in \mathcal{O}_X(X)$, then each component of $Z(f_1, \ldots, f_t)$ has dimension $>$ dim $X - t$. (That is, the codimension of each componenet is $\leq t$).*

Proof. Induction on t. □

Remark 0.2. There are lots of examples when dim $Z(f_1, \ldots, f_t) >$ dim $X - t$. For example, polynomials that were independent in a larger ring could become dependent in the quotient ring.

Example 0.3. Recall the twisted cubic in $\mathbf{P}^3$ described by

$$\gamma : \mathbf{P}^1 \to \mathbf{P}^3 \; : \; \gamma[s,t] = [s^3, s^2t, st^2, t^3] = [x_0, x_1, x_2, x_3].$$

Then

$$\Gamma = Im(\gamma) = \left\{ [x_0, x_1, x_2, x_3] \in \mathbf{P}^3 \; : \; \det{}_2 \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix} = 0 \right\}$$

Thus we can see that three quadrics (the three 2-dimensinal determinants) will cut out the (1-dimensional) twisted cubic.

Does the converse of our theorem hold?

Theorem 0.4. *Let X be an affine variety, Z a subvariety of X (closed, irreducible) of codimension t. Then there exist a finite number of functions $f_1, \ldots, f_t \in \mathcal{O}_X(X)$ such that Z is a component of $Z(f_1, \ldots, f_t)$.*

Proof. First let $t = 1$. Then $Z \subset X$ has codimension 1. Let $R = \mathcal{O}_X(X)$. Then $0 \neq I(Z) \subseteq R$ is a prime ideal of R. Choose $0 \neq f \in I(Z)$. Since f vanishes on Z, $Z \subseteq Z(f)$. But now Z has codimension 1, and $Z(f)$ has pure codimension 1 by Krull's theorem. It follows by a dimension argument that we must have Z as a component of $Z(f)$.

Now suppose that $t \geq 2$. Let $0 \neq f \in I(Z)$. Then $Z(f) = Z_1 \cup \cdots \cup Z_r$, Z_i the irreducible componenets of Z. Each of these components has codimension 1 by assumption. We know that codim$(Z, X) \geq 2$. It follows that we have $Z \subseteq Z(f)$, but $Z_i \not\subseteq Z$ by a dimension argument. In particular, $I(Z) \not\subseteq I(Z_i)$ for any i. Hence there exists some $g \in I(Z) \backslash I(Z_1) \cup \cdots \cup I(Z_r)$, as the $I(Z_i)$ are prime ideals. Thus we have found a function g that vanishes on Z but not on Z_i for any i. Each component of the zero locus $Z(f, g)$ is of pure codimension 2, however:

$$Z \subseteq Z(f,g) = (Z_1 \cap \{g = 0\}) \cup \cdots \cup (Z_r \cap \{g = 0\})$$

where $Z_i \cap \{g = 0\} \subsetneq Z_i$. Iterate this process to find functions $f_1, \ldots, f_t \in I(Z)$ such that $Z(f_1, \ldots, f_t)$ is of pure codimension t and $Z \subseteq Z(f_1, \ldots, f_t)$. Hence Z is one of the irreducible components of $Z(f_1, \ldots, f_t)$. □

Definition 0.5. Let X be a variety. Then $Z \subseteq X$, $\operatorname{codim}(Z, X) = t$ is a *local complete intersection* if for all $p \in Z$, there exists some affine open neighborhood $U \subseteq X$ containing p and $f_1, \ldots, f_t \in \mathcal{O}_X(X)$ such that $Z \cap U = \{x \in U \ : \ f_1(x) = \cdots = f_t(x) = 0\}$.

[Note: we proved before that Z was a component of this locus.]

Remark 0.6. Even in codimension 1 there are subvarieties which are not local complete intersections.

Theorem 0.7. *If X is an affine variety such that $A(X) = \mathcal{O}_X(X)$ is a Unique Factorization Domain, then every codimension 1 subvariety of X is of type $Z(f)$ for some $f \in \mathcal{O}_X(X)$.*

Remark 0.8. Let $X = \mathbb{A}^n$. Then $A(X) = k[x_1, \ldots, x_n]$ is a UFD. Hence every codimension 1 subvariety is a hypersurface.

Example 0.9. $X = \mathbb{A}^3$, $C = \{(t^3, t^4, t^5) \ : \ t \in \mathbb{C}\}$. Then C cannot be cut out by two equations.

Proof. Suppose $R = \mathcal{O}_X(X)$ is a UFD. Then every nonzero minimal prime ideal is principal. To see that this is true, suppose $0 \neq \mathfrak{P} \subseteq R$ is a minimal prime ideal of R. Let $0 \neq f \in \mathfrak{P}$. Take f' to be any prime factor of f such that $f' \in \mathfrak{P}$. Then (f') is a prime ideal of R contained in $\mathfrak{P}$. Then it follows by the minimality of $\mathfrak{P}$ that $(f') = \mathfrak{P}$, so $\mathfrak{P}$ is principal as desired.

Now let $Z \subset X$ be of pure codimension 1, with $Z = Z_1 \cup \cdots \cup Z_r$, where each Z_i is of codimension 1 in X. The Z_i are maximal irreducible sets in X, and thus correspond in a 1-1 manner with the minimal prime ideals in $A(X)$, which recall are all principal. It follows that $I(Z_i) = (f_i)$, and thus $I(Z) = (f_1 f_2 \cdots f_r)$, a principal ideal as desired. □

1. Smooth Varieties

To get a feel for smooth varieties, think about which varieties we want to consider smooth. Consider the varieties

$$X = \{x^2 + y^2 = 1\}, \ Y = \{y^2 = x^3 + x^2\}, \ Z = \{y^2 = x^3\}.$$

The first of these is a circle, which clearly has a neighborhood of each $p \in X$ which is isomorphic to $\mathbb{C}$. Thus we want X to be smooth. Y, however, has a point where the function is not 1-1 - i.e., the image of the function in affine space crosses itself. Thus this function is not smooth. Finally, Z has a cusp - a point where it is not differentiable as we think of it in standard calculus. We want Z to also not be smooth.

Definition 1.1. Let $X \subseteq \mathbb{A}^n$ be an affine variety and let $p \in X$. Then $I(X) = \langle f_1, \ldots, f_t \rangle$, where the f_i are generators of $I(X)$. Define the *Jacobian matrix at p* by

$$J_p = \left(\frac{\partial f_i}{\partial x_j}(p) \right), i = 1, \ldots, t; j = 1, \ldots, n.$$

We say that p is a *smooth point* if $\operatorname{Rank}(J(p)) = n - d$, where $d = \dim X$.

Remark 1.2. We always have $t \geq n - d$. But it is not true that you can always pick $n - d$ generators.

Example 1.3. Let

$$X^1 = \left\{ \begin{array}{c} x + y + z = 0 \\ x^2 + 2y + z^3 = 0 \end{array} \right. \subseteq \mathbb{C}^3.$$

Let $p = (0, 0, 0)$. Then

$$J_0(X) = \left(\begin{array}{ccc} 1 & 1 & 1 \\ 0 & 2 & 0 \end{array} \right)$$

It is clear that $\mathrm{Rank}(J(X)_0) = 2$, hence X is smooth at p. [I.e., X is locally isomorphic to $\mathbb{C}$.] To see the local isomorphism, define a map from $X \to \mathbb{C}$ by

$$(x, y, z) \mapsto x.$$

This function is a local isomorphism. The fact that this is so is given by the implicit function theorem, which tells us that given $f_1(x, y, z) = 0$, $f_2(x, y, z) = 0$, we can plug in $y = y(x)$, $z = z(x)$ and solve about the point $(0, 0, 0)$.

Definition 1.4. If R is a Noetherian local ring over k with maximal ideal $\mathfrak{M}$, we say that R is *regular* if $\dim \mathfrak{M}/\mathfrak{M}^2 = \dim R$, the Krull dimension of R.

Example 1.5. Note that we always have $\dim \mathfrak{M}/\mathfrak{M}^2 \geq \dim R$.

Theorem 1.6. *Let $X \subseteq \mathbb{A}^n$ be an affine variety. Then X is smooth at the point $p \in X$ if and only if $\mathcal{O}_{X,p}$ is a regular local ring. I.e. if and only if we have dim $\mathfrak{M}_p/\mathfrak{M}_p^2 =$ dim $\mathcal{O}_{X,p} =$ dim X.*

Definition 1.7. If $(X, \mathcal{O}_X)$ is an abstract variety, and $p \in X$ then X is smooth at p if and only if $\mathcal{O}_{X,p}$ is regular. I.e., $\dim \mathfrak{M}_p/\mathfrak{M}_p^2 = \dim X$.