Mar. 4, Friday (1.5 hr class)
(NoteTEXer: Paul Fili)

1. Smooth Varieties (cont.)

Recall our setup from last time: let $X^d \subseteq \mathbb{A}^n$ be an affine variety with $I(X) = (f_1, \ldots, f_t)$. We say $p \in X$ is a *smooth* (or *nonsingular* or *regular*) point, which we denote $p \in X_{\text{reg}}$, if $\operatorname{rank}\left(\frac{\partial f_\alpha}{\partial x_j}\right)_{\substack{1\le\alpha\le t\\ 1\le j\le n}} = n-d$. (We take the opportunity here to note that this is *formal* differentiation of polynomials.) Algebraic geometry since Grothendieck has been about developing invariant concepts that work in a wide variety of cases. We want concepts that work for general algebraic varieties and also (later) for schemes. When we develop schemes, we will tackle deformation problems about how varieties vary in families. So let us make a more general definition:

Definition 1.1. We say a variety X is *smooth* at a point $p \in X$ if $\mathcal{O}_{X,p}$ is a regular local ring. (Recall $\mathcal{O}_{X,p}$ is a regular local ring if $\dim_k \mathfrak{m}_p/{\mathfrak{m}_p}^2 = \dim \mathcal{O}_{X,p} = \dim X$).

This is a definition that does not depend on the embedding of X or the choice of generators. We now show that these two definitions are, in fact, equivalent in the affine case:

Proposition 1.2. *Let $X^d \subseteq \mathbb{A}^n$ as above. Then* $\operatorname{rank}\left(\frac{\partial f_\alpha}{\partial x_j}\right)_{\substack{1\le\alpha\le t\\ 1\le j\le n}} = n-d$ *if and only if $\mathcal{O}_{X,p}$ is a regular local ring.*

Proof. Let $p = (a_1, \ldots, a_n) \in X$, and denote by $\mathfrak{a}_p = (x_1-a_1, \ldots, x_n-a_n) \subseteq k[x_1, \ldots, x_n]$ the maximal ideal corresponding to p in $\mathbb{A}^n$. We wish to identify $\mathfrak{a}_p/{\mathfrak{a}_p}^2$ with the vector space of hyperplanes in $\mathbb{A}^n$ passing through the origin. Define the map:

$$\theta : \mathfrak{a}_p/{\mathfrak{a}_p}^2 \xrightarrow{\sim} k^n$$

$$\theta(f) = \left\langle \frac{\partial f}{\partial x_1}(p), \ldots, \frac{\partial f}{\partial x_n}(p) \right\rangle.$$

Again we note that this is formal differentiation of polynomials, and all of the old rules apply. In particular Leibnitz's rule tells us that for $f, g \in \mathfrak{a}_p$, we have $\theta(fg) = 0$. So for each f we have an associated hyperplane though 0 given by $\sum \frac{\partial f}{\partial x_i}(p) x_i = 0$. Note that the $\theta(x_i)$ form a basis for k^n. So let $I(X) \subseteq R = k[x_1, \ldots, x_n]$ and $I(X) = (f_1, \ldots, f_r)$, then we claim that $\operatorname{rank} J = \dim \theta(I(X))$ where $J = \left(\frac{\partial f_\alpha}{\partial x_j}\right)_{\substack{1\le\alpha\le t\\ 1\le j\le n}}$ is the Jacobian. To see this, note that $\theta(f_i)$ span $\theta(I(X))$, and that $\dim \theta(I(X))$ is indeed the rank of the matrix formed by the $\theta(f_i)$, which is J. Then we have the isomorphism:

$$\begin{aligned}\theta(I(X)) &\cong I(X)/(I(X) \cap \mathfrak{a}_p{}^2) \\ &\cong \frac{I(X) + \mathfrak{a}_p{}^2}{\mathfrak{a}_p{}^2}\end{aligned}$$

Recall that $\mathcal{O}_{X,p} = (R/I(X))_{\widetilde{\mathfrak{a}}_p}$ where $\widetilde{\mathfrak{a}}_p$ is the maximal ideal in the quotient corresponding to $\mathfrak{a}_p$. Now $\mathfrak{m}_p \subseteq \mathcal{O}_{X,p}$. Note that $R_\mathfrak{m}$ is of course a local ring and $\widetilde{\mathfrak{m}} = \mathfrak{m}R \subset R_\mathfrak{m}$ is its maximal ideal, and we have a natural isomorphism $\widetilde{\mathfrak{m}}/\widetilde{\mathfrak{m}}^2 \cong \mathfrak{m}/\mathfrak{m}^2$. So

$$\mathfrak{m}_p/\mathfrak{m}_p{}^2 \cong \widetilde{\mathfrak{a}}_p/\widetilde{\mathfrak{a}}_p^2 \cong \frac{\mathfrak{a}_p + I(X)}{\mathfrak{a}_p{}^2 + I(X)} \cong \frac{\mathfrak{a}_p}{\mathfrak{a}_p{}^2 + I(X)}$$

since $I(X) \subseteq \mathfrak{a}_p$. Now note that we have

$$\mathfrak{a}_p{}^2 \subseteq \mathfrak{a}_p{}^2 + I(X) \subseteq \mathfrak{a}_p$$

so

$$\dim \frac{\mathfrak{a}_p}{\mathfrak{a}_p{}^2 + I(X)} + \dim \frac{\mathfrak{a}_p{}^2 + I(X)}{\mathfrak{a}_p{}^2} = \dim \mathfrak{a}_p/\mathfrak{a}_p{}^2 = n$$

since we showed that $\theta : \mathfrak{a}_p/\mathfrak{a}_p{}^2 \xrightarrow{\sim} k^n$ and $\dim k^n = n$. So we have

$$\dim \mathfrak{m}_p/\mathfrak{m}_p{}^2 + \operatorname{rank}(J) = n$$

and therefore

$$\dim \mathfrak{m}_p/\mathfrak{m}_p{}^2 = d \iff \operatorname{rank}(J) = n - d$$

so our two definitions are in fact equivalent. □

2. The Zariski Tangent Space

Definition 2.1. Let $p \in X$. Then we define the *Zariski tangent space* $T_p(X)$ to be

$$T_p(X) = \left(\mathfrak{m}_p/\mathfrak{m}_p{}^2\right)^*.$$

What we proved above was that X is smooth at p iff $\dim T_p(X) = \dim X$. Note that in general $\dim \mathfrak{m}_p/\mathfrak{m}_p{}^2 \geq \dim X$ with equality iff p is a smooth point of X, so in particular $\dim T_p(X) \geq \dim(X)$.

Remark 2.2. The set $X_{\text{sing}} = \{x \in X : X \text{ is smooth at } x\}^c$ (note the complement) is a proper Zariski closed subset in X, that is, $X_{\text{reg}} = \{x \in X : X \text{ is smooth at } x\}$ is an open dense subset.

Since the singularity of a point is a local matter, we might as well assume X is affine. Let $0 = p \in X \subseteq \mathbb{A}^n$, $I(X) \subseteq k[x_1, \dots, x_n]$. Define

$$k[x_1, \dots, x_n]^1 = \{a_1x_1 + \cdots + a_nx_n : a_i \in k\}.$$

We can always decompose $f = f^{(0)} + f^{(1)} + \cdots + f^{(d)}$ where $d = \deg f$ and the $f^{(i)}$ are homogenous of degree i. Then we define

$$I(X)^1 = \{f^{(1)} : f \in I(X)\}$$

to the be linear part of the ideal.

Proposition 2.3. *There exists an isomorphism*

$$\eta : \mathfrak{m}_p/\mathfrak{m}_p{}^2 \xrightarrow{\sim} k[x_1, \ldots, x_n]^1/I(X)^1$$

so $T_p(X) \cong Z(I(X)^1) = \{x \in \mathbb{A}^n : f^1(x) = 0 \ \forall f \in I(X)\}$. *Since we can write*

$$f = \underbrace{\sum_i \frac{\partial f}{\partial x_i}(p) x_i}_{f^{(1)}} + f^{(2)} + \cdots + f^{(d)},$$

we can write the tangent space as

$$T_p(X) = \{x \in \mathbb{A}^n : \sum_i \frac{\partial f}{\partial x_i}(p) x_i = 0 \ \forall f \in I(X)\}.$$

Proof. Let $\phi \in \mathfrak{m}_p$. Define

$$\eta(\phi) = \sum_{i=1}^n \frac{\partial \phi}{\partial x_i}(p) x_i \mod I(X).$$

Recall that $\phi = \dfrac{f(x_1, \ldots, x_n)}{g(x_1, \ldots, x_n)}$ for some $f, g \in A(X)$, $g(p) \neq 0$. Note that f, g are not uniquely determined as polynomials in $k[x_1, \ldots, x_n]$, in fact they are only defined up to $f + I(X)$. This is why we must mod out by $I(X)$ to make η well-defined. Note that

$$\eta^{-1}\left(\sum a_i x_i\right) = \sum a_i x_i \mod \mathfrak{m}_p^2$$

so η is indeed the desired isomorphism, and the result for the tangent space follows. □

Let's do some examples.

Example 2.4. Consider the curve $X : y = x^2 - x$ in $\mathbb{A}^2$, and let $p = (0, 0)$. Then $T_p(X) = \{y + x = 0\}$, since $f(x, y) = y - x^2 + x$, we have $f^{(1)} = y + x$. Note in particular that $\dim T_p X = 1$ so $p = (0, 0)$ and $p \in X_{\text{reg}}$.

Example 2.5. Consider the nodal curve $X : y^2 = x^2 + x^3$, and let $p = (0, 0)$. Then $f(x, y) = y^2 - x^2 - x^3$, and we have $f^{(1)} = 0$, so $T_p(X) = \mathbb{A}^2$. Therefore X is singular at p, but X is smooth everywhere else.

Example 2.6. Consider the cuspidal curve $X : y^2 = x^3$, and let $p = (0, 0)$. Then $f(x, y) = y^2 - x^3$, and we have $f^{(1)} = 0$, so again $T_p(X) = \mathbb{A}^2$, and X is singular at p.

There is a whole branch of algebraic geometry devoted to singularity theory. The simplest possible singularity is a node.

Example 2.7. A marginally worse singularity is given in the tacnode curve $X : x^2 = x^4 + y^4$. Let $p = (0, 0)$, and note once again that $T_p(X) = \mathbb{A}^2$, and X is singular at p.

Example 2.8. Consider now the Fermat hypersurface, $X = \{[x_0, \ldots, x_n] \in \mathbb{P}^n : {x_0}^d + \cdots + {x_n}^d =\}$. Then $\dfrac{\partial f}{\partial x_i} = d\,{x_i}^{d-1}$. Suppose now that the characteristic of our underlying field $\operatorname{char} k = d$ (is a prime). Then all of the partials vanish and the Fermat hypersurface is singular. **Caution:** Keep this in mind, and remember that $\frac{d}{dx}x^p = 0$ in characteristic p.

Remark 2.9. There is yet another description of the Zariski tangent space. We can define $T_p(X)$ to be the vector space of derivations $D : \mathcal{O}_{X,p} \to k$, that is, the space of k-linear functions that satisfy the Leibnitz rule, $D(fg) = f(p)D(g) + g(p)D(f)$. To see this, note that $T_p(X) = \operatorname{Hom}_k(\mathfrak{m}_p/{\mathfrak{m}_p}^2, k)$. So consider the map $(\ell : \mathfrak{m}_p/{\mathfrak{m}_p}^2 \to k) \rightsquigarrow D_\ell(f)$ where $D_\ell(f) = \ell(f - f(p))$. We must check that the Leibnitz rule is satisfied by D_ℓ, but that easily follows since ℓ annihilates the square, ${\mathfrak{m}_p}^2$, as we remarked above.

2.1. **Smooth Projective Varieties.** It is useful to have a projective Jacobian criterion. Suppose $X \subseteq \mathbb{P}^n$ is a projective variety, and $p \in X$. Let $I(X) = (f_1, \ldots, f_t)$ be the homogenous ideal in $n+1$ variables. Then X is smooth at p if and only if

$$\operatorname{rank}(J) = \operatorname{rank}\left(\frac{\partial f_\alpha}{\partial x_i}(p)\right)_{\substack{\alpha=1,\ldots,t\\ i=0,\ldots,n}} = n - \dim X.$$

This is just an application of the Euler rule for homogenous polynomials, $\sum x_i \dfrac{\partial F}{\partial x_i} = d \cdot F$ where $d = \deg F$, which implies that if $F(p) = 0$ for some $p \in X$ then $\sum x_i \dfrac{\partial F}{\partial x_i} = 0$ as well at p.

Example 2.10. We will show that the twisted cubic $X \subseteq \mathbb{P}^3$, which we recall is given by the vanishing of the 2×2 minors of the following matrix:

$$X : \det\nolimits_2 \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix} = 0,$$

which is the vanishing of the quadratics

$$\begin{aligned} Q_1 &= x_0x_2 - {x_1}^2 \\ Q_2 &= x_0x_3 - x_1x_2 \\ Q_3 &= x_1x_3 - {x_2}^2 \end{aligned}$$

Then we have

$$J = \begin{pmatrix} x_2 & -2x_1 & x_0 & 0 \\ x_3 & -x_2 & -x_1 & x_0 \\ 0 & x_3 & -2x_2 & x_1 \end{pmatrix}.$$

We claim that $\operatorname{rank}(J) = 2$. If J had rank 1 then we could check the 2×2 minors of J, for example

$$\begin{vmatrix} x_3 & -x_2 \\ 0 & x_3 \end{vmatrix} = 0 \implies x_3 = 0$$

and similarly we conclude that $x_1 = 0$, and then we must have $x_0 = x_2 = 0$, but this is impossible. Therefore the twisted cubic is indeed smooth.

2.2. **Differentials of Morphisms.** Suppose $f : X \to Y$ is a morphism. Then there is a k-linear map:

$$df_p = T_p(f) : T_p(X) \to T_{f(p)}Y$$

where we recall that $T_p(X) = (\mathfrak{m}_p/{\mathfrak{m}_p}^2)^*$ and $T_{f(p)} = (\mathfrak{m}_{f(p)}/{\mathfrak{m}_{f(p)}}^2)^*$. To see this, recall that we have an induced map of the local rings

$$f_p^\# : \mathcal{O}_{Y,f(p)} \to \mathcal{O}_{X,p}$$

given by the pullback of functions, and therefore

$$f_p^\#(\mathfrak{m}_{f(p)}) \subseteq \mathfrak{m}_p \qquad \text{and} \qquad f_p^\#({\mathfrak{m}_{f(p)}}^2) \subseteq {\mathfrak{m}_p}^2$$

so we have an induced map:

$$f_p^\# : \mathfrak{m}_{f(p)}/{\mathfrak{m}_{f(p)}}^2 \to \mathfrak{m}_p/{\mathfrak{m}_p}^2$$

and then the dual of this map gives us $df_p = T_p(f) = (f_p^\#)^*$, the desired map.

How do we describe the tangent map locally? Suppose we have a map $f : X \to Y$ where $X \subseteq \mathbb{A}^n$, $Y \subseteq \mathbb{A}^m$ and $f = (f_1, \ldots, f_m)$. Assume for simplicity that $p = 0$ is the origin. Then we showed that the $x_1, \ldots x_n$ span our space of hyperplanes through the origin in $\mathbb{A}^n$, so we write this as vector space as $k\langle x_1, \ldots, x_n\rangle$, and $T_p(X) \subseteq T_p(\mathbb{A}^n)$ and we write

$$T_p(\mathbb{A}^n) = (k\langle x_1, \ldots, x_n\rangle)^* = k\left\langle \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right\rangle$$

where

$$\frac{\partial}{\partial x_i} : k^n \to k$$

is the dual to x_i. Then

$$T_p(X) = \left\{ v = \sum a_i \frac{\partial}{\partial x_i} : \forall f \in I(X),\ \sum a_i \frac{\partial f}{\partial x_i}(p) = 0 \right\}$$

and we have the map

$$\begin{array}{c} T_p(X) \subseteq k\left\langle \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right\rangle \\ \Big\downarrow{\scriptstyle T_p(f)} \\ T_p(Y) \subseteq k\left\langle \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_m}\right\rangle \end{array}$$

$T_p(f)$ is the linear map with matrix

$$\left(\frac{\partial f_\alpha}{\partial x_i}(p)\right)_{\substack{i=1,\ldots,n \\ \alpha=1,\ldots,m}}.$$

This is how we obtain the tangent map, just as we did in differential geometry.

In calculus, we had an inverse function theorem when the differential was an isomorphism. *The inverse function theorem fails in an algebraic setting,* which isn't really a surprise. Even if we had a morphism $f : X \to Y$ and $p \in X$ such that $T_p(f) : T_p(X) \xrightarrow{\sim} T_p(Y)$, to require that an open neighborhood be isomorphic would be a *very* strong condition, in fact it would make X and Y birationally isomorphic. This is much different than the situation in analysis.

Example 2.11. Let $X : y = x^2 \subset \mathbb{A}^2$. Let $\pi : X \to \mathbb{A}^1$ be the projection of the y coordinate. Then we have the map

$$T_{(x,x^2)}(\pi) : T_{(x,x^2)}(X) \to T_{x^2}(\mathbb{A}^1)$$
$$\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) \mapsto \frac{\partial}{\partial y}$$

which is an isomorphism for all $x \neq 0$. Take for example $p = (1, 1)$. Then $T_p(\pi)$ is an ismorphism. In analysis this tells us that $x = \sqrt{y}$, $(x, y) \mapsto y$ is an isomorphism. We claim that $\nexists \varnothing \neq U \subseteq X$ open such that $\pi|_U$ is an isomorphism. Clearly, in the Zariski topology any such U will contain lots of pairs (x, x^2) and $(-x, x^2)$, and therefore the map is not one-to-one.

But it's a still a special condition for $T_p(f)$ to be an isomorphism, and we therefore make the following definition:

Definition 2.12. $f : X \to Y$ is *étale* at p if $T_p(f)$ is an isomorphism.

There is a way to say when the map is étale using local rings.