Algebraic Geometry (Notes taken by John Meth)

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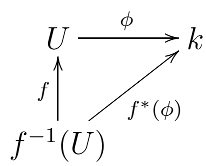
Let $X \subseteq \mathbb{A}^n$ be an irreducible algebraic set and $\mathcal{O}_X$ its structure sheaf of regular functions.

Let $X \subseteq \mathbb{A}^n$, $Y \subseteq \mathbb{A}^m$ be irreducible algebraic sets, $f : X \to Y$ any continuous map. Then for any open set $U \subseteq Y$ and for any function $\phi : U \to k$, we define

$$f^*(\phi) := \phi \circ f : f^{-1}(U) \to k.$$

That is, define $f^*(\phi)$ so that the following diagram commutes:



Theorem 1.1. *Let $X \subseteq \mathbb{A}^n$, $Y \subseteq \mathbb{A}^m$ be irreducible algebraic sets and $f : X \to Y$ a continuous map. Then the following are equivalent:*

1 f is a morphism.

2 $\forall U \subseteq Y$ open,

$$\begin{array}{rcl} f^* : \mathcal{O}_Y(U) & \to & \mathcal{O}_X(f^{-1}(U)) \\ \phi & \mapsto & \phi \circ f \end{array}$$

is a ring homomorphism (that is, f^* maps regular functions to regular functions).

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$$\begin{array}{rcl} f^* : \mathcal{O}_Y(Y) & \to & \mathcal{O}_X(X) \\ || & & || \\ A(Y) & & A(X) \end{array}$$

is a ring homomorphism.

Proof. $1 \Rightarrow 2$: Suppose $f : X \to Y$ is a morphism, so $f = (f_1, \ldots, f_m)$, $f_i \in k[x_1, \ldots, x_n]$. Let $\phi : U \to k$, $\phi \in \mathcal{O}_Y(U)$, so $f^*\phi = \phi \circ f : f^{-1}(U) \to k$. Pick $x \in f^{-1}(U)$, then $f(x) \in U$ so $\exists g,\, h \in A(Y)$, $\phi = g/h$, and $h(f(x)) \neq 0$. Now $f^*(\phi) = \phi(f_1, \ldots, f_m) = \frac{g(f_1,\ldots,f_m)}{h(f_1,\ldots,f_m)}$, where $h(f_1, \ldots, f_m)(x) = h(f(x)) \neq 0$, so $\frac{g(f_1,\ldots,f_m)}{h(f_1,\ldots,f_m)} \in \mathcal{O}_X(f^{-1}(U))$. The fact that it is a ring homomorphism is immediate.

$2 \Rightarrow 3$: Obvious if we set $U = Y$.

$3 \Rightarrow 1$: Suppose $f^* : A(Y) \to A(X)$ is a ring homomorphism. We've seen that this uniquely determines $f : X \to Y$ as follows: $f(x) = (f^*y_1, \ldots, f^*y_m)(x)$, so f is a morphism from X to Y.

□

Definition 1. *An affine variety over k, an algebraically closed field, is a topological space X together with a sheaf $\mathcal{O}_X$ of k-valued functions on X such that $(X, \mathcal{O}_X)$ is isomorphic to an irreducible algebraic set together with its structural sheaf.*

Remark 1.2. *$(X, \mathcal{O}_X)$ is an affine variety $\Leftrightarrow \exists Y \subseteq \mathbb{A}^n$, an irreducible algebraic set, such that $(X, \mathcal{O}_X) \cong (Y, \mathcal{O}_Y)$, i.e. there exists a homeomorphism $f : X \to Y$, such that for all open sets $U \subseteq Y$, the pullback map $f_U^* : \mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}(U))$ is an isomorphism of rings and these isomorphisms f_U^* are compatible with restrictions.*

Note that if $X \subseteq \mathbb{A}^n$ is an irreducible algebraic set, then $(X, \mathcal{O}_X)$ is an affine variety, but there are many more examples of affine varieties.

Theorem 1.3. *Suppose $(X, \mathcal{O}_X)$ is an affine variety and $f \in A(X) = \mathcal{O}_X(X)$. Then the distinguished open set $(X_f, \mathcal{O}_{X_f})$ is an affine variety.*

X_f is open in X, since $X_f = X - Z(f)$, so for $U \subseteq X_f$ open, U is open in X and $\mathcal{O}_{X_f}(U) = \mathcal{O}_X(U)$.

Example 1. *Before we start to prove this theorem, we will explain the idea of the proof in a simple example. We consider the distinguished open set $\mathbb{A}^1 - \{(0,0)\} = \mathbb{A}^1_x$ and we realize this as an affine variety in $\mathbb{A}^2$. Let*
$Z(xy = 1) = V \subseteq \mathbb{A}^2$. Then projection $\pi : V \to \mathbb{A}^1_x$, $\pi(x, y) = x$, is an isomorphism from V to $\mathbb{A}^1_x$, thus $\mathbb{A}^1_x$ is an affine variety.

Proof. Suppose $X \subseteq \mathbb{A}^n$, and let $\mathfrak{a} = I(X)$ be the ideal of X. For $f \in A(X)$, let $\tilde{f} \in k[x_1, \ldots, x_n]$ be a representative for f, i.e. $\tilde{f} \mapsto f \bmod \mathfrak{a}$. We want to realize X_f as an affine variety in $\mathbb{A}^{n+1}$. We define the ideal

$$J = \mathfrak{a} + (1 + \tilde{f}(x_1, \ldots, x_n)x_{n+1}) \subseteq k[x_1, \ldots, x_{n+1}].$$

Claim: J is a prime ideal. Since

$$\frac{k[x_1, \ldots, x_{n+1}]}{J} = \frac{k[x_1, \ldots, x_n]}{\mathfrak{a}}\left[\frac{1}{f}\right] = A(X)_f$$

is a localization of the integral domain $A(X)$, the ring $A(X)_f$ must also be a domain. Thus J is prime.

Define

$$\begin{array}{rcl} & \pi & \\ Z(J) & \to & \mathbb{A}^n \\ (x_1, \ldots, x_{n+1}) & \mapsto & (x_1, \ldots, x_n). \end{array}$$

Then $\pi : Z(J) \to X_f$ is a homeomorphism. Note that

$$\begin{gathered} (x_1, \ldots, x_{n+1}) \in Z(J) \Leftrightarrow \\ (x_1, \ldots, x_{n+1}) \in Z(\mathfrak{a}) \subseteq \mathbb{A}^{n+1} \text{ and } \tilde{f}(x_1, \ldots, x_n)x_{n+1} = 1 \Leftrightarrow \\ (x_1, \ldots, x_n) \in X \text{ and } \tilde{f}(x_1, \ldots, x_n) \neq 0 \Leftrightarrow \\ (x_1, \ldots, x_n) \in X_f \end{gathered}$$

hence π establishes a bijection between $Z(J)$ and X_f. Proving π is a homeomorphism is left to the reader.

Claim: π establishes an isomorphism $(Z(J), \mathcal{O}_{Z(J)}) \cong (X_f, \mathcal{O}_{X_f})$. We need to show that $\forall U \subseteq X_f$ open, $\pi^* : \mathcal{O}_{X_f}(U) \to \mathcal{O}_{Z(J)}(\pi^{-1}(U))$ is an isomorphism. It is enough to check this for the distinguished open subsets of X_f, $U = X_g \cap X_f$, $g \in A(X)$. Fix $g \in A(X)$, then $U = X_g \cap X_f = X_{fg}$, and

$$\begin{aligned}\mathcal{O}_{X_f}(U) &= \mathcal{O}_X(X_{fg}) \\ &= A(X)_{fg} \\ &= \frac{k[x_1, \ldots, x_n]}{\mathfrak{a}}\left[\frac{1}{fg}\right].\end{aligned}$$

Let $\tilde{g}$ be a representative for g, then $\pi^{-1}(U) = \pi^{-1}(X_g) = Z(J)_{\tilde{g}}$, and

$$\begin{aligned}\mathcal{O}_{Z(J)}(\pi^{-1}(U)) &= \mathcal{O}_{Z(J)}(Z(J)_{\tilde{g}}) \\ &= \mathcal{O}_{Z(J)}(Z(J))_{\tilde{g}} \\ &= \frac{k[x_1, \ldots, x_{n+1}]}{J}\left[\frac{1}{\tilde{g}}\right] \\ &= \frac{k[x_1, \ldots, x_n]}{\mathfrak{a}}\left[\frac{1}{f}\right]\left[\frac{1}{g}\right] \\ &= \frac{k[x_1, \ldots, x_n]}{\mathfrak{a}}\left[\frac{1}{fg}\right]\end{aligned}$$

Thus $\pi : Z(J) \to X_f$ gives isomorphisms $\pi^* : \mathcal{O}_{X_f}(U) \to \mathcal{O}_{Z(J)}(\pi^{-1}(U))$, so $(X, \mathcal{O}_{X_f}) \cong (Z(J), \mathcal{O}_{Z(J)}) \Rightarrow (X_f, \mathcal{O}_{X_f})$ is an affine variety.

□

Corollary 1.4. *Every affine variety X has a basis for the Zariski topology consisting of open affine subsets, $\{X_f | f \in A(X)\}$.*

Remark 1.5. *Although for X affine and $f \in A(X)$, the open set X_f is affine, it is not true that any open subset of an affine variety is affine.*

Example 2. *Let $X = \mathbb{A}^2$, $U = \mathbb{A}^2 - \{(0, 0)\}$. Then we have shown that U is not affine, although U is the union of the open affine sets $\mathbb{A}^2_x := \mathbb{A}^2 - \{x = 0\}$ and $\mathbb{A}^2_y := \mathbb{A}^2 - \{y = 0\}$.*