Morphisms (Notes taken by Zachary Miner)

Let $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^n$ be irreducible algebraic sets. A morphism $\alpha : X \to Y$ is defined as

$$\alpha(x_1, \ldots, x_n) = (f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n))$$

where $f_i \in k[x_1, \ldots, x_n]$, induces a morphism $\alpha^* : \mathcal{A}(Y) \to \mathcal{A}(X)$ between the affine coordinate rings of Y and X defined by

$$\alpha^*(\bar{\phi}(y_1, \ldots, y_n)) = \phi(f_1, \ldots, f_m)$$

where $\phi(f_1, \ldots, f_m)$ is seen to be a polynomial in $x_1, \ldots, x_n$.

So α^* is a morphism of k-algebras, and we will soon prove that α^* determines α, which says that thinking about α or about α^* in the end amounts to the same thing. This shows that the affine coordinate ring of an affine variety (which is an algebraic object), completely determines the geometry of the variety. To show this, we look at how α^* acts on each y_i:

$$\alpha^*(y_i) = f_i \bmod I(X)$$

(determined up to the ideal $I(X)$) gives

$$\alpha^*(y_i) : Y \to k$$

and $f_i : X \to k$ is well-defined, so that we recover α:

$$\alpha(x) = (\alpha^*(y_1), \ldots, \alpha^*(y_m))(x) \ \in \mathbb{A}^m.$$

Claim: $\alpha(X) \subseteq Y$.

We need to show that $\alpha(X)$ is annihilated by every element of $I(Y)$. Pick $g \in I(Y)$ We need to show that $g(\alpha(x)) = 0$. This is equivalent with showing that

$$g(\alpha^*(y_1), \ldots, \alpha^*(y_m))(x) = 0 \Leftrightarrow \alpha^*(g(y_1, \ldots, y_m)) = 0$$

(α^* is a ring homomorphism). But $\alpha^*(I(Y)) \subseteq I(X)$, which means

$$\alpha(g(y_1, \ldots, y_m)) \in I(X) \Leftrightarrow \alpha^*(g(y_1, \ldots, y_m))|_X = 0.$$

Thus $\alpha : X \to Y$ is determined by α^*.

We have proved:

Theorem 0.1. *There is a 1-1 correspondence between homomorphisms of algebraic sets and homomorphisms of k-algebras:*

$$\mathrm{Hom}_{\text{alg. sets}}(X, Y) \longleftrightarrow \mathrm{Hom}_{k-\text{alg}}(\mathcal{A}(Y), \mathcal{A}(X))$$

where

$$(\alpha : X \to Y) \longleftrightarrow (\alpha^* : \mathcal{A}(Y) \to \mathcal{A}(X))$$

So, we have a way to tell when two algebraic sets are isomorphic: check if their corresponding affine coordinate rings are isomorphic.

Example 0.2. Let $Y = \mathbb{A}^1$ so that $\mathcal{A}(Y) = k[t]$. What is $\mathrm{Hom}(X, \mathbb{A}^1)$?

$$\{f : X \to k\} = \mathrm{Hom}(X, \mathbb{A}^1) = \mathrm{Hom}_{k-\mathrm{alg}}(k[t], \mathcal{A}(X)) \cong \mathcal{A}(X)$$

$\mathrm{Hom}_{k-\mathrm{alg}}(k[t], \mathcal{A}(X))$ is determined uniquely by where t goes:

$t \rightsquigarrow u \in \mathcal{A}(X)$. (Affine coordinate ring of X) $\longleftrightarrow$ (morphisms from X to $\mathbb{A}^1$).

Now recall that we have defined morphisms between two algebraic sets and we prove that they are continuous in the Zariski topology:

Proposition 0.3. *Let $X \subseteq \mathbb{A}^n$, $Y \subseteq \mathbb{A}^m$, and let $\alpha : X \to Y$ be a morphism. Then α is continuous in the Zariski topology.*

Proof. First, $\alpha = (f_1, \ldots, f_m)$. We need to show that if $Z \subseteq Y$ is closed, then $\alpha^{-1}(Z)$ is also closed. Now, $I(Z) =< h_1, \ldots, h_s >$, where $h_i \in \mathcal{A}(Y)$. So,

$$\begin{aligned} \alpha^{-1}(Z) &= \{x \in X : h_1(\alpha(x)) = \ldots = h_s(\alpha(x)) = 0\} \\ &= \{x \in X : h_1(f_1(x), \ldots, f_m(x)) = 0, \ldots, h_s(f_1(x), \ldots, f_m(x)) = 0\} \end{aligned}$$

And $\alpha^{-1}(Z)$ is thus seen to be Zariski closed in X. Therefore, α is continuous. □

Example 0.4. Let L be the line $5x+3y = 2 \subset \mathbb{A}^2$. It should be that $L \cong \mathbb{A}^1$. To prove this, we will show their affine coordinate rings are isomorphic: Now,

$$\mathcal{A}(L) \cong \mathcal{A}(\mathbb{A}^1),$$

is true since

$$\frac{k[x, y]}{(5x + 3y - 2)} \cong k[t],$$

so it follows that $L \cong \mathbb{A}^1$.

Example 0.5. $X : (y^2 = x^3) \subset \mathbb{A}^2$. In you draw this curve over $\mathbb{R}$, you see that it has two branches meeting at the point $(0, 0)$ which is a cusp. We define the morphism $f : \mathbb{A}^1 \to X$, by $f(t) = (t^2, t^3)$. f is seen to be bijective, hence a homeomorphism in the Zariski topology. Now, we claim that f is not an isomorphism:

$$f^* : \mathcal{A}(X) \to k[t] \ \text{ with } \ \mathcal{A}(X) = \frac{k[x, y]}{(x^2 - y^3)}$$

and since $f^*(\phi(x, y)) = \phi(t^2, t^3)$ it is clear that f^* cannot be surjective, as $t \notin \mathrm{Im}(f^*)$. So, $X \ncong \mathbb{A}^1$, and f is an example of a homeomorphism which i s not an isomorphism.

Example 0.6. (**Frobenius**) Let $k = \bar{k}$ and $\mathrm{char}(k) = p > 0$. Consider $F : \mathbb{A}^1 \to \mathbb{A}^1$ defined by $F(t) = t^p$. Then $F^* : k[t] \to k[t]$ is the induced morphism that sends

$$(a_0 + a_1 t + \ldots + a_s t^s) \rightsquigarrow (a_0 + a_1 t^p + \ldots + a_s t^{ps}).$$

F is another example of a homeomorphism that is not an isomorphism (the fixed points of F are the points in $\mathbb{F}_p$).

Moral: Morphisms of algebraic sets are more than just continuous functions. (This info will be contained in sheafs.) So, we will need more than the Zariski topology to describe morphisms.

Definition 0.7. (**Localization**) Let R be a ring, and let $S \subset R$ be a multiplicative set, $(0 \notin S)$.

$$S^{-1}R := \{\frac{a}{b} : a \in R,\ b \in S\}$$

Define an equivalence relation $\sim$ on this set by:

$$(a, b) \sim (c, d) \Leftrightarrow \exists s \in S \text{ such that } s(ad - bc) = 0.$$

Remark 0.8. If R is a domain, and $\mathfrak{p}$ is a prime ideal, let $S = R - \mathfrak{p}$. Then

$$\begin{aligned} S^{-1}R &= \{\tfrac{x}{y} : x \in R,\ y \notin \mathfrak{p}\} \\ &= \text{localization of } R \text{ at } \mathfrak{p}. \end{aligned}$$

In particular, $R_{(0)} = Q(R) =$ the fraction field of R

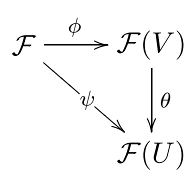
(Sheaves are very widespread in math. Their definition makes sense in any topological space.)

Definition 0.9. Let X be a topological space. A *presheaf* $\mathcal{F}$ on X consists of the following data:

i. $\forall$(open)$U \subseteq X$ we associate a set $\mathcal{F}(U) = \Gamma(U, \mathcal{F})$ (called sections of $\mathcal{F}$ over U)

ii. For open sets $U \subseteq V \subseteq X$(we associate restriction maps) $\text{res}_{V,U} : \mathcal{F}(V) \to \mathcal{F}(U)$ satisfying the following axioms:

1. $res_{U,U} : \mathcal{F}(U) \to \mathcal{F}(U)$ is the identity map $1_{\mathcal{F}(U)}$
2. Compatibility: If $U \subseteq V \subseteq W$, then the following diagram commutes:



where $\phi = res_{W,V}$, $\psi = res_{W,U}$, $\theta = res_{V,U}$. In other words, $res_{W,U} = res_{V,U} \circ res_{W,V}$

Example 0.10. Let $X = \mathbb{R}^n$ with the Euclidean topology, $\mathcal{F}$ be a presheaf of continuous functions. So for each open set $U \subset \mathbb{R}^n$, we have that

$$\mathcal{F}(U) = \mathcal{C}(U, \mathbb{R}) = \{f : U \to \mathbb{R}, \text{ such that } f \text{ is continuous}\}.$$

If $U \subseteq V \subseteq \mathbb{R}^n$ then $\mathcal{F}(V) \to \mathcal{F}(U)$ (that is, $(f : V \to \mathbb{R}) \to (f|_U : U \to \mathbb{R})$) is defined to be the ordinary restriction map. It is left as an exercise to check that the axioms are satisfied.

Example 0.11. Let $X = \mathbb{R}^n$, and let $\mathcal{G}$ be the pre-sheaf of $\mathcal{C}^\infty$-differentiable function. Then $\mathcal{G}(U) = \{f : U \to \mathbb{R} :\ f \text{ is } \mathcal{C}^\infty\}$

Example 0.12. Let $X = \mathbb{C}$. Then
$$U \rightsquigarrow \mathcal{T}hol(U) = \{f : U \to \mathbb{C} : \ f \text{ is holomorphic}\}.$$