**Course:** Topics in Stochastic Analysis: BSDE
**Term:** Spring 2012
**Instructor:** Gordan Zitkovic

# Lecture 2
## THE LIPSCHITZ THEORY

The goal of this lecture is to define a class of BSDEs and to study their existence and uniqueness properties under Lipschitz or linear-growth-type conditions on the driver. We assume throughout the lecture that the filtration is Brownian, i.e., generated by a $d$-dimensional Brownian motion. We also pick and fix an integer $m \in \mathbb{N}$ - a dimension of the process $\{Y_t\}_{t\in[0,T]}$ defined below. Finally, a **time horizon** $T > 0$ is picked.

**Remark 2.1** In order to fit into the notation introduced in Lecture 1, we assume that all processes are defined on $[0,\infty)$, but their values are set to 0 (for integrators) or kept constant (for semimartingales) after $T$. This way, for example, one inherits the right notion of a local martingale on $[0,T]$, namely, the one which requires the stopping times to *stabilize* at 0 (as opposed to merely converge there).

**Definition 2.2** *Let $\xi \in \mathbb{L}^2_m(\mathcal{F}_T, \mathbb{R}^m)$ be a random vector, and let $g : \Omega \times [0,T] \times \mathbb{R}^m \times \mathbb{R}^d \to \mathbb{R}^m$ be a $(\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^n) \otimes \mathcal{B}(\mathbb{R}^{m\times d}), \mathcal{B}(\mathbb{R}^m))$-measurable function, where $\mathcal{P}$ denotes the progressive $\sigma$-algebra on $\Omega \times [0,T]$.*

*A pair of processes $(Y,Z) \in \mathcal{S}^2_m \times \mathfrak{h}^2_{m\times d}$ is said to be a **solution to a BSDE** $(g,\xi)$ if*

$$Y_t = \xi + \int_t^T g(u, Y_u, Z_u)\, du - \int_t^T Z_u\, dB_u, \text{ for all } t \in [0,T], \text{ a.s.} \tag{2.1}$$

**Remark 2.3**

(1) It is, implicitly, a part of the definition that the first integral in (2.1) converges for all $t$, a.s.
(2) The function $f$ is called the **driver** of the BSDE $(f,\xi)$ and the random variable $\xi$ its **terminal condition**.

## 2.1. Lipschitz Drivers

The existence proof in the Lipschitz case is rather standard; it rephrases the BSDE as a fixed point and uses Banach's fixed-point theorem. It appears that the good choice of a function space is $\mathcal{S}^2_m \times \mathfrak{h}^2_{m\times d}$, equipped with a *weighted* norm

$$||(Y,Z)||^2_\beta = ||w_\beta Y||^2_{\mathcal{S}^2_m} + ||w_\beta Z||^2_{\mathfrak{h}^2_{m\times d}} = \mathbb{E}\left[\sup_{t\in[0,T]} e^{\beta t}\left|Y_t\right|^2\right] + \mathbb{E}\left[\int_0^t e^{\beta t}\left|Z_t\right|^2 dt\right],$$

where $w_\beta(t) = e^{\frac{1}{2}\beta t}$; the parameter $\beta > 0$ will be determined in the course of the proof. It is standard to see that $||\cdot||_\beta$ is indeed a norm, under which $\mathcal{S}^2_m \times \mathfrak{h}^2_{m\times d}$ is a Banach space.

Our first task is to establish the existence in a special, in a sense trivial, case.

**Proposition 2.4 (Existence and estimates in the "trivial" case)** *Suppose that there exists a process* $\{G_t\}_{t\in[0,T]} \in \mathscr{h}^2_m$ *such that*

$$g(\omega, t, y, z) = G_t(\omega), \text{ for all } \omega, t, y, z.$$

*Then, the BSDE* $(g, \xi)$ *admits a unique solution* $(Y, Z)$ *and there exists a constant* $C$*, which depends only on* $T$*,* $m$ *and* $d$*, such that*

$$||(Y,Z)||^2_\beta \leq C\left(e^{\beta T}||\xi||^2_{\mathbb{L}^2_m} + \tfrac{1}{\beta}\mathbb{E}\left[\int_0^T e^{\beta u}\left|G_u\right|^2\,du\right]\right). \tag{2.2}$$

Proof. (*Note:* The letter $C$ will denote a generic constant which depends only on $T$, $m$ or $d$, and may change from line to line.)

Let the martingale $\{M_t\}_{t\in[0,T]}$ be defined by

$$M_t = \mathbb{E}\left[\zeta|\mathcal{F}_t\right] - \mathbb{E}\left[\zeta\right],\ t \in [0,T], \text{ where } \zeta = \xi + \int_0^T G_u\,du,$$

so that, thanks to the conditions $\xi \in \mathbb{L}^2_m$ and $G \in \mathscr{h}^2_m$, we have $M \in \mathcal{H}^2_m$. According to the martingale-representation theorem, there exists $Z \in \mathscr{h}^2_{m\times d}$ such that

$$M_t = \int_0^t Z_u\,dB_u, \text{ for } t \in [0,T],$$

Consequently, the pair $(Y, Z)$, where $\{Y\}_{t\in[0,T]}$ is given by $Y_t = M_t - \int_0^t G_u\,du$ satisfies

$$Y_t + \int_t^T Z_u\,dB_u = \xi + \int_t^T G_u\,du.$$

Since $G \in \mathscr{h}^2_m$, we have $\zeta \in \mathbb{L}^2_m$; hence $M$ is a square-integrable martingale. By the maximal inequality, we have $||\,|M|^*_T\,||_{\mathbb{L}^2_m} \leq C||\xi||_{\mathbb{L}^2_m}$. At the same time, $\left|\int_t^T G_u\,du\right| \leq (\int_0^T G_u^2\,du)^{1/2} \in \mathbb{L}^2_m$, and so $Y \in \mathcal{S}^2_m$, i.e., $|Y|^*_T \in \mathbb{L}^2$. Consequently, $(Y, Z)$ is a solution of the BSDE $(g, \xi)$, as described in Definition 2.2.

Next, we prove the estimate (2.2); thanks to linearity uniqueness will follow directly from it. We start by applying Itô's formula to the process $\{e^{\beta t}\left|Y_t\right|^2\}_{t\in[0,T]}$ on $[t, T]$, and remind the reader that $(\cdot)^\tau$ denotes transposition:

$$e^{\beta T}\left|\xi\right|^2 = e^{\beta t}\left|Y_t\right|^2 + \int_t^T \beta e^{\beta u}\left|Y_u\right|^2\,du + 2\int_t^T e^{\beta u}Y_u^\tau\Big(Z_u\,dB_u - G_u\,du\Big) + \int_t^T e^{\beta u}\left|Z_u\right|^2\,du. \tag{2.3}$$

Using the simple inequality $-\beta y^2 + 2yg \leq \frac{1}{\beta}g^2$, valid for all $y, g \in \mathbb{R}$ and $\beta > 0$, we get

$$e^{\beta t}\left|Y_t\right|^2 + \int_t^T e^{\beta u}\left|Z_u\right|^2\,du \leq e^{\beta t}\left|\xi\right|^2 + \tfrac{1}{\beta}\int_t^T e^{\beta u}\left|G_u\right|^2\,du + 2(N_T - N_t), \tag{2.4}$$

where $N_t = \int_0^t e^{\beta u}Y_u^\tau Z_u\,dB_u$. Taking a supremum over $t \in [0,T]$ yields

$$\sup_{t\in[0,T]} e^{\beta t}\left|Y_t\right|^2 + \int_0^t e^{\beta u}\left|Z_u\right|^2\,du \leq e^{\beta T}\left|\xi\right|^2 + \tfrac{1}{\beta}\int_0^T \left|G_u\right|^2\,du + 4N^*_T. \tag{2.5}$$

By the BDG inequality and the fact that $Cab \leq \frac{1}{4}a^2 + C^2b^2$, we get

$$\mathbb{E}\left[N_T^*\right] \leq C\mathbb{E}\left[\sqrt{\langle N\rangle_T}\right] = C\mathbb{E}\left[\sqrt{\int_0^T e^{2\beta u}\left|Y_u^\tau Z_u\right|^2\,du}\right]$$

$$\text{(2.6)} \qquad \leq C\mathbb{E}\left[\Big(\sup_{t\in[0,T]} e^{\beta u}\left|Y_u\right|^2\Big)^{1/2}\Big(\int_0^T e^{\beta u}\left|Z_u\right|^2\,du\Big)^{1/2}\right]$$

$$\leq \tfrac{1}{4}\mathbb{E}\left[\sup_{t\in[0,T]} e^{\beta t}\left|Y_2\right|^2\right] + C^2\mathbb{E}\left[\int_0^T e^{\beta u}\left|Z_u\right|^2\,du\right]$$

We have already established that that $\sup_{t\in[0,T]}\left|Y_u\right|^2 \in \mathbb{L}^1$, so we can combine (2.5) and (2.6) to obtain

$$\text{(2.7)} \qquad ||(Y,Z)||_\beta^2 \leq C\left(e^{\beta T}||\xi||_{\mathbb{L}^2_m} + \tfrac{1}{\beta}\mathbb{E}\left[\int_0^T e^{\beta u}\left|G_u\right|^2\,du\right] + \mathbb{E}\left[\int_0^T e^{\beta u}\left|Z_u\right|^2\,du\right]\right)$$

On the other hand, since $M \in \mathcal{H}_1^1$, we have $\mathbb{E}\left[M_T - M_t\right] = 0$ and so, by taking the expectation in (2.4) and disregarding $|Y_0|$, we obtain the following estimate:

$$\text{(2.8)} \qquad \mathbb{E}\left[\int_0^T e^{\beta u}\left|Z\right|_u^2\,du\right] \leq e^{\beta T}\mathbb{E}\left[\left|\xi\right|^2\right] + \tfrac{1}{\beta}\mathbb{E}\left[\int_0^T e^{\beta u}\left|G_u\right|^2\,du\right]$$

Finally, (2.2) follows from (2.7) and (2.8). ■

We are now ready to state a prove our central existence-and-uniqueness theorem for BSDE with Lipschitz generators. Hypothesis 2.5 lists a standard set of assumptions under which our method works:

**Hypothesis 2.5 (Lipschitz continuity)**

(1) $\xi \in \mathbb{L}^2_m$,
(2) $g(\cdot,\cdot,0,0) \in \hbar^2_m$, and
(3) there exists a constant $K \geq 0$ such that

$$|g(\omega,t,y_2,z_2) - g(\omega,t,y_1,z_1)| \leq K\Big(\,|y_2 - y_1| + |z_2 - z_1|\,\Big),$$

for all $(\omega,t) \in \Omega \times [0,T]$ and all $y_1, y_2 \in \mathbb{R}^m$, $z_1, z_2 \in \mathbb{R}^{m\times d}$.

**Theorem 2.6 (Existence with Lipschitz Drivers)** *Under Hypothesis 2.5, the BSDE $(g,\xi)$ admits a unique solution.*

PROOF. For $(\hat{Y},\hat{Z}) \in \mathcal{S}^2_m \times \hbar^2_{m\times d}$, let $(Y,Z)$ be the unique solution to the BSDE

$$Y_t = \xi + \int_t^T g(t,\hat{Y}_t,\hat{Z}_t)\,du - \int_t^T Z_u\,dB_u,$$

Thanks to (the last two parts of) Hypothesis 2.5, we have

$$\left|g(t,\hat{Y}_t,\hat{Z}_t)\right| \leq |g(t,0,0)| + \left|g(t,\hat{Y}_t,\hat{Z}_t) - g(t,0,0)\right|^2 \leq |g(t,0,0)| + K\big(|\hat{Y}_t| + |\hat{Z}_t|\big) \in \hbar^2_1.$$

Therefore, Proposition 2.4 applies and the map $\Phi$, given by

$$\Phi(\hat{Y},\hat{Z}) = (Y,Z)$$

maps $\mathcal{S}^2_m \times \hbar^2_{m\times d}$ into itself.

Next, we pick $(Y^{(1)}, Z^{(1)})$ and $(Y^{(2)}, Z^{(2)})$ in $\mathcal{S}^2_m \times \hbar^2_{m\times d}$, and note the their difference $(\delta Y, \delta Z) = (Y^{(2)} - Y^{(1)}, Z^{(2)} - Z^{(1)})$ solves the BSDE

$$\delta Y_t = 0 + \int_t^T G_u\, du - \int_t^T \delta Z_u\, dB_u, \text{ where } G_t = g(t, Y^{(1)}_t, Z^{(1)}_t) - g(t, Y^{(2)}_t, Z^{(2)}_t).$$

The estimate (2.2) now implies that

$$\begin{aligned} ||\Phi(Y^{(1)}, Z^{(1)}) - \Phi(Y^{(2)}, Z^{(2)})||^2_\beta &\leq \tfrac{C}{\beta}\mathbb{E}\left[\int_0^T e^{\beta u}\left|g(t, Y^{(1)}_t, Z^{(1)}_t) - g(t, Y^{(2)}_t, Z^{(2)}_t)\right|^2 du\right] \\ &\leq \tfrac{CK^2}{\beta}||(\delta Y, \delta Z)||^2_\beta. \end{aligned}$$

If we choose $\beta = 2CK^2$, the map $\Phi$ becomes a $||\cdot||_\beta$-contraction on the Banach space $\mathcal{S}^2_m \times \hbar^2_{m\times d}$. Consequently, by the Banach fixed-point theorem, it admits a unique fixed point $(Y, Z)$. It follows now immediately from the definition of the map $\Phi$, that $(Y, Z)$ is the unique solution to the BSDE $(g, \xi)$. ■

In addition to uniqueness, the fact that the Banach fixed-point theorem applies yields another interesting property of the solutions of BSDEs:

**Proposition 2.7 (Picard Iterations)** *Given a pair $(g, \xi)$ which satisfies Hypothesis 2.5, let $(Y^{(i)}, Z^{(i)})$, $i \in \mathbb{N}_0$ be a sequence in $\mathcal{S}^2_m \times \hbar^2_{m\times d}$ defined as follows*

- $Y^{(0)} = 0$ *and* $Z^{(0)} = 0$;
- $(Y^{(i+1)}, Z^{(i+1)})$ *is the solution of the BSDE*

$$Y^{(i+1)}_t = \xi + \int_t^T g(u, Y^{(i)}_u, Z^{(i)}_u)\, du - \int_t^T Z^{(i+1)}_u\, dB_u.$$

*Then, we have*

$$\sup_{t\in[0,T]}\left|Y^{(i)}_t - Y_t\right| + \sqrt{\int_0^T \left|Z^{(i)}_u - Z_u\right|^2 du} \to 0, \text{ a.s., in and } \mathbb{L}^2,$$

*as well as $Z^{(i)} \to Z$, $dt \otimes d\mathbb{P}$-a.e.*

Proof. The $\mathbb{L}^2$-convergence follows directly from the convergence in the Banach fixed-point theorem. One also gets the a.s.-convergence by observing that

$$\sum_{i=0}^\infty ||(Y^{(i+1)}, Z^{(i+1)}) - (Y^{(i)}, Z^{(i)})||_\beta < \infty,$$

for large-enough $\beta > 0$, and that the norms $||\cdot||_\beta$ and $||\cdot||_0$ are uniformly equivalent. ■

The following estimate, reminiscent of that in Proposition 2.4, now follows immediately from the above considerations:

**Corollary 2.8 (Well-posdeness of BSDEs)** *Under Hypothesis 2.5, the solution $(Y, Z)$ of the BSDE $(g, \xi)$ satisfies*

$$\mathbb{E}[\sup_{t\in[0,T]} |Y_t|^2] + \mathbb{E}[\int_0^T |Z_u|^2\, du] \leq C\Big(||\xi||^2_{\mathbb{L}^2_m} + ||g(t,0,0)||^2_{\mathcal{H}^2_m}\Big),$$

*where the constant $C$ depends only on $m$, $d$ and $T$.*

## 2.2. **Dimension** 1

In the case $m = 1$, one can prove existence of a solution to the BSDE $(g, \xi)$ under considerably weaker hypotheses. The difference between dimension 1 and higher dimensions, is that in the 1-dimensional case, we have the so-called *comparison principle* on our disposal. Note how only one of the drivers needs to be Lipschitz; the other is quite general.

**Theorem 2.9 (A comparison principle)** *Let $(Y, Z)$ and $(Y', Z')$ be solutions to the BSDEs $(g, \xi)$ and $(g', \xi')$, respectively. Suppose that*

- $m = 1$,
- *$(g, \xi)$ satisfy the conditions of Hypothesis 2.5,*
- $g'(t, Y'_t, Z'_t) \in \mathcal{H}^2_1$.

*and that*

$$\xi \leq \xi',\ \text{a.s., and } g(t,y,z) \leq g'(t,y,z),\ \text{for all } (t,y,z) \in [0,T]\times\mathbb{R}\times\mathbb{R}^d,\ \text{a.s.} \tag{2.9}$$

*Then, $Y_t \leq Y'_t$, for all $t \in [0, T]$, a.s.*

Proof. We set

$$\delta_t = Y_t - Y'_t,\ \ I_t = \mathbf{1}_{\{Y_t \geq Y'_t\}},\ \ \Delta_t = g(t, Y_t, Z_t) - g'(t, Y'_t, Z'_t) \text{ and } D_t = e^{\beta t}\big(\delta_t^+\big)^2,$$

so that, by the Itô-Tanaka formula, we get

$$dD_t = \beta \delta_t^2 I_t\, dt + 2e^{\beta t} I_t \delta_t\, d\delta_t + e^{\beta t} I_t\, d\langle \delta\rangle_t = e^{\beta t}\Big(R_t\, dt + dM_t\Big)$$

where

$$R_t = \beta I_t \delta_t^2 - 2I_t\delta_t\Delta_t + I_t(Z_t - Z'_t)^2 \text{ and } M_t = 2\int_0^t \delta_u I_u (Z_u - Z'_u)\, dW_u.$$

We first note that $M \in \mathcal{H}^1_1$; this the fact that $\delta \in \mathcal{S}^2_1$ and $Z - Z' \in \mathcal{H}^2_1$. Next, we observe that on $Y_t \geq Y'_t$ we have

$$\begin{aligned}\delta_t\Delta_t &= (Y_t - Y'_t)^+\Big(g(t, Y_t, Z_t) - g(t, Y'_t, Z'_t) + g(t, Y'_t, Z'_t) - g'(t, Y'_t, Z'_t)\Big)\\ &\leq K(Y_t - Y'_t)^+\Big(\left|Y_t - Y'_y\right| + \left|Z_t - Z'_t\right|\Big).\end{aligned}$$

and so

$$R_t \geq I_t\Big((\beta - 2K)\delta^2 - 2K\delta\left|Z' - Z_t\right| + \left|Z_t - Z'_t\right|^2\Big),$$

which is nonnegative if $\beta$ is large enough ($\beta \geq (K+1)^2$). It follows that, for large enough $\beta$, $D_t$ is a submartingale with $D_T = 0$. Therefore, $D_t \leq 0$, for all $t \in [0, T]$, a.s., i.e., $Y_t \leq Y'_t$, for all $t \in [0, T]$, a.s. ∎

The proof of the comparison principle above relies on the Itô-Tanaka formula, which is available only in dimension 1. That alone does not preclude the possibility of having a comparison principle in higher dimensions. What does preclude it, though, is the following example: One application of the comparison principle is the following existence result for solutions to one-dimensional BSDE whose drivers have at-most linear growths, but are not necessarily Lipschitz continuous. An approximation via so-called **inf-convolution**, described in the next proposition, is the main idea behind the proof.

**Proposition 2.10 (Regularizing properties of the inf-convolution)** *Let $g:\mathbb{R}^d\to\mathbb{R}$ be a continuous function of at-most linear growth, i.e., such that $g(x)\leq C(1+|x|)$, for some $C\geq 0$. For $n\in\mathbb{N}$, $n>C$, we define*

$$g_n(x)=\inf_{\xi\in\mathbb{R}^d}\big(g(\xi)+n\,|x-\xi|\,\big).$$

*Then*

*(1) The sequence $\{g_n\}_{n\in\mathbb{N}}$ is non-decreasing and $|g_n(x)|\leq C(1+|x|)$, for all $x\in\mathbb{R}^d$, $n\in\mathbb{N}$.*
*(2) $g_n$ is Lipschitz with the Lipschitz constant at most $n$.*
*(3) If $\{x_n\}_{n\in\mathbb{N}}\to x$, then $g_n(x_n)\to g(x)$, (i.e., $g_n\to g$, locally uniformly).*

PROOF. Without loss of generality, we assume that $C<1$ so that the sequence $g_n$ takes only finite values for all $n\in\mathbb{N}$. It is clear that $g_n(x)\leq g_{n+1}(x)$ for all $x$, $n\in\mathbb{N}$, so that

$$C(1+|x|)\geq g(x)\geq g_n(x)=\inf_{\xi\in\mathbb{R}^d}\big(g(\xi)+n\,|x-\xi|\,\big)\geq\inf_{\xi\in\mathbb{R}}\big(-C(1+|\xi|)+n\,|\xi-x|\,\big)=-C(1+|x|),$$

and (1) follows. To show (2), we simply note that, $g_n$ is an infimum of a family of Lipschitz functions with Lipschitz constant $n$; it follows that $g_n$ is itself Lipschitz with Lipschitz constant at most $n$.

Finally, for (3), we define take a sequence $\{x_n\}_{n\in\mathbb{N}}$ with $x_n\to x$, and assume, contrary to the statement, that $g_n(x_n)\not\to g(x)$. By the upper semicontinuity of $g$, we have

$$g(x)\geq\limsup_n g(x_n)\geq\limsup_n g_n(x_n),$$

so that, without loss of generality, we can assume that $\{x_n\}_{n\in\mathbb{N}}$ is such that

$$g_n(x_n)\to h=g(x)-\delta,$$

for some $\delta\in(0,\infty]$.

For $n\in\mathbb{N}$, let $\xi_n$ be such that $g_n(x_n)\geq g(\xi_n)+n\,|x_n-\xi_n|-\delta/2$. Thanks to the linear growth of $g$, the sequence $\{\xi_n\}_{n\in\mathbb{N}}$ is bounded; without loss of generality, we assume that it converges towards some $\xi\in\mathbb{R}^d$. The function $g$ is lower semicontinuous, which leads to the following contradiction

$$g(\xi)\leq\liminf_n g(\xi_n)\leq\liminf_n\Big(g_n(x_n)-n\,|x_n-\xi_n|-\delta/2\Big)\leq\begin{cases}-\infty, & \xi\neq x,\\ g(\xi)-\delta/2, & x=\xi.\end{cases}$$ ■

The set of hypothesis under which one-dimensional BSDEs have solutions is given below:

**Hypothesis 2.11 (Linear growth)**

(1) $\xi\in\mathbb{L}^2_m$,
(2) $g(\cdot,\cdot,0,0)\in h^2_m$, and

(3) $(y,z) \mapsto g(\omega,t,z,y)$ is continuous, $dt \otimes d\mathbb{P}$-a.e.

(4) there exists a constant $K \geq 0$ such that

$$|g(\omega,t,y,z)| \leq |g(\omega,t,0,0)| + K\big(\,|y| + |z|\,\big),$$

for all $(\omega,t) \in \Omega \times [0,T]$ and all $y \in \mathbb{R}^m$, $z \in \mathbb{R}^{m\times d}$.

**Theorem 2.12 (Existence with $m = 1$ and linear growth)** *Suppose that $m = 1$ and let the BSDE $(g,\xi)$ satisfy the conditions of Hypothesis 2.11. Then, $(g,\xi)$ admits a minimal and a maximal solution.*

Proof (Sketch). Set $b(y,z) = K(|y| + |z|)$, and denote by $U^+$ and $U^-$ the solutions of the BSDEs $(b,\xi)$ and $(-b,\xi)$. For $n \in \mathbb{N}$, let the function $g_n$ be given by the inf-convolution of $g$ and $n\,|\cdot|$, i.e.,

$$g_n(\omega,t,y,z) = \inf_{(\xi_y,\xi_z)\in\mathbb{R}\times\mathbb{R}^d} \Big( g(\omega,t,\xi_y,\xi_z) + n\,|(\xi_y,\xi_z) - (y,z)|\,\Big).$$

Proposition 2.10 applies to the sequence $g_n$, $dt \otimes d\mathbb{P}$-a.e., and we will use its conclusions without special mention. In particular, solutions $(Y^{(n)}, Z^{(n)})$ to the BSDE $(g_n,\xi)$ exist in the appropriate spaces. Due to the comparison principle (Theorem 2.9), the sequence $\{Y^{(n)}\}_{n\in\mathbb{N}}$ is pointwise monotone and $U^- \leq Y^{(n)} \leq U^+$. From there, one shows that the sequences $\{Y^{(n)}\}_{n\in\mathbb{N}}$ and $\{Z^{(n)}\}_{n\in\mathbb{N}}$ are Cauchy in $\mathcal{h}^2_1$ and $\mathcal{h}^2_d$, and that the limits $Y$, and $Z$ satisfy the limiting BSDE. While the above are the main ideas, there are several highly nontrivial subtleties involved in the proof; we point the reader to the original paper [LSM97]. ■

## References

[LSM97] J. P. Lepeltier and J. San Martin, *Backward stochastic differential equations with continuous coefficient*, Statist. Probab. Lett. **32** (1997), no. 4, 425–430.