

Quaternionic Geometry of Everything

Tamás Hausel
Department of Mathematics
University of Texas at Austin

April 8, 2004
Colloquium at Rice University

"An electric circuit seemed to close; and a spark flashed forth the herald (as I foresaw immediately) of many long years to come of definitely directed thought and work by myself, if spared, and, at all events, on the part of others if I should even be allowed to live long enough distinctly to communicate the discovery. Nor could I resist the impulse - unphilosophical as it may have been - to cut with a knife on a stone of Brougham Bridge, as we passed it, the fundamental formula which contains the Solution of the Problem, but, of course, the inscription has long since mouldered away."

[William Rowan Hamilton in 1843, on his invention of quaternions]

Skew field of Quaternions

$$\mathbb{H} := \{a + bi + cj + dk \mid a, b, c, d \in \mathbb{R}\}$$

$$i^2 = j^2 = k^2 = ijk = -1$$

↓

$$ij = -ji = k$$

$$jk = -kj = i$$

$$ki = -ik = j$$

“In general, although in one sense I hope that I am actually growing modest about the quaternions, from my seeing so many peeps and vistas into future expansions of their principles, I still must assert that this discovery appears to me to be as important for the middle of the nineteenth century as the discovery of fluxions was for the close of the seventeenth.”

[William Rowan Hamilton in 1853, ten years after his discovery of quaternions]

As the theory of fluxions, today called calculus, has been used universally in mathematics and the sciences, is there such a thing, as Hamilton claimed, as

“Quaternionic Geometry of Everything” ?

Hyperkähler manifolds

- A *hyperkähler manifold* is a Riemannian manifold of dimension $4n$ with holonomy contained in

$$Sp(n) \subset SU(2n) \subset U(2n) \subset SO(4n).$$

Roughly speaking it is a Riemannian manifold, so that the metric is compatible with a quaternionic structure on the tangent bundle of the manifold

- Another definition is to say, that the manifold is Kähler with respect to three Kähler structures (I, ω_I, g) , (J, ω_J, g) and (K, ω_K, g) corresponding to the same Riemannian metric g , and the complex structures, as endomorphisms of the tangent bundle which satisfy the quaternionic relationship

$$I^2 = J^2 = K^2 = IJK = -1.$$

A hyperkähler manifold is thus a

- Riemannian manifold
- Complex manifold
- Kähler manifold
- Calabi-Yau manifold
- Ricci-flat (Einstein) manifold
- Symplectic manifold
- Holomorphic symplectic manifold
- Usually a quasi-projective variety
- Sometimes an affine variety

Perhaps we could consider the variety of geometries appearing in hyperkähler geometry as the

“Quaternionic Geometry of Everything” ?

Examples of hyperkähler manifolds

- Flat examples: \mathbb{H} and \mathbb{H}^n
- a non-compact example: the Calabi metric (1979) on $T^*\mathbb{C}\mathbb{P}^n$
- compact examples of Beauville (1983): X is a K3 surface or an abelian surface, then the Hilbert scheme $X^{[n]}$ of n -points on X is a hyperkähler manifold of dimension $4n$
- two more compact examples due to O'Grady (1999,2003) in dimensions 12 and 20

String theories enter

- Since the 1970's Superstring theory claims to be "The Theory of Everything"
- Superstring theories with large amount of supersymmetry, like those with $N=4$ supersymmetry, are the easiest to test mathematically
- $N=4$ supersymmetry implies that certain moduli spaces in the theory will acquire hyperkähler structure
- Predictions of duality theories, when applied to $N=4$ supersymmetric theories, yield conjectures on the geometry of these hyperkähler manifolds

Perhaps the study of these mathematical tests in duality theories of $N=4$ supersymmetric string theories, could be called the

“Quaternionic Geometry of
Everything”

Mathematicians enter

- In 1978 Atiyah-Drinfeld-Hitchin-Manin constructed all self-dual Yang-Mills instantons on \mathbb{R}^4 using quaternions.
- Donaldson (1982) showed that the space of the above ADHM solution space has the natural structure of a hyperkähler manifold
- Many more moduli spaces of solutions to physically relevant PDE's turned out to have hyperkähler structure (Atiyah-Hitchin, Hitchin, Kronheimer-Nakajima, ...)

Zoo

Hodge theory on hyperkähler manifolds

- Certain S-Duality conjectures in certain $N = 4$ supersymmetric string theories predict the dimension of the space of L^2 harmonic forms on some of these non-compact hyperkähler manifolds
- Most of these conjectures are still open due to the difficulty in understanding the asymptotics of the metrics on these spaces and to the considerable analytical obstacles for Hodge theory on these non-compact hyperkähler manifolds
- Nevertheless some results for checking these S-duality conjectures were recently achieved by Hitchin (1999) and Hausel-Hunsicker-Mazzeo (2002)

Hodge Theory in a nutshell

- On a compact oriented Riemannian manifold M^n , the *Hodge* $*$ operator is defined:

$$\begin{aligned} * : \Omega^k(M) &\rightarrow \Omega^{n-k}(M) \\ (*\alpha) \wedge \beta &= \langle \alpha, \beta \rangle_g \text{Vol}_g, \end{aligned}$$

we say that a form $\alpha \in \Omega^k$ is *harmonic* iff

$$d\alpha = 0, \quad d(*\alpha) = 0$$

- The *Hodge theorem* asserts that every De Rham cohomology class contains a unique harmonic representative. I.e. the solutions to the above PDE are governed by the topology, namely the cohomology of the manifold as

$$H_{DR}^*(M) \cong H_{sing}^*(M, \mathbb{R}).$$

Hard Lefschetz

- If M^{2n} is a compact Kähler manifold, with Kähler form $\omega \in \Omega^2(M)$, then a non-trivial consequence of the above Hodge theorem is the Hard Lefschetz theorem, which asserts that the map

$$\begin{aligned} L^k : H_{DR}^{n-k}(M) &\rightarrow H_{DR}^{n+k}(M) \\ L^k([\alpha]) &:= [\omega^k \wedge \alpha] \end{aligned}$$

is an isomorphism

- One consequence of this is Poincaré duality, i.e. that $H_{DR}^{n-k}(M) \cong H_{DR}^{n+k}(M)$. Consequently Hard Lefschetz cannot be true for non-compact manifolds, where Poincaré duality fails.

Weak-Hard Lefschetz

Let M^{4n} be a hyper-compact hyperkähler manifold (examples are Nakajima's quiver varieties, toric hyperkähler varieties and the moduli space of Higgs bundles), then the De Rham cohomology above the middle degree $2n$ vanishes, and for the Kähler form ω_I , Hausel (2003) proves that

$$\begin{aligned} L^k &: H_{DR}^{n-k}(M) \rightarrow H_{DR}^{n+k}(M) \\ L^k([\alpha]) &:= [\omega_I^k \wedge \alpha] \end{aligned}$$

is an injective map.

Application to Combinatorics

- Bielawski-Dancer (2000) introduced toric hyperkähler manifolds, which are the hyperkähler analogues of toric varieties. They found that their topology is governed by the combinatorics of some affine hyperplane arrangement. In particular the Betti numbers of the toric hyperkähler variety encode the face vector of the bounded complex of the corresponding affine hyperplane arrangement. Classification of the possible face vectors of hyperplane arrangements is an unsolved problem in combinatorics.
- Hausel-Sturmfels (2002) used the Weak-Hard Lefschetz theorem above to get hitherto unknown inequalities among the face numbers of hyperplane arrangements.

Number theory enters

One example of an affine hyperkähler variety is the $GL(n, \mathbb{C})$ character variety $\mathcal{M}_B(GL(n, \mathbb{C}))$ of a compact Riemann surface Σ of genus $g > 1$. It is defined as the space of twisted representations of the fundamental group of Σ into $GL(n, \mathbb{C})$ modulo conjugation:

$$\mathcal{M}_B(GL(n, \mathbb{C})) := \{A_1, B_1, \dots, A_g, B_g \in GL(n, \mathbb{C}) \mid A_1^{-1} B_1^{-1} A_1 B_1 \dots A_g^{-1} B_g^{-1} A_g B_g = \xi_n Id\} / GL(n, \mathbb{C})$$

The strategy of Hausel-Rodriguez-Villegas (2003) for getting the Betti numbers of $\mathcal{M}_B(GL(n, \mathbb{C}))$ is to count the rational points of the variety over a finite field \mathbb{F}_q . Thus we have to count points of

$$\mathcal{M}_B(GL(n, \mathbb{F}_q)) := \{A_1, B_1, \dots, A_g, B_g \in GL(n, \mathbb{F}_q) \mid A_1^{-1} B_1^{-1} A_1 B_1 \dots A_g^{-1} B_g^{-1} A_g B_g = \xi_n Id\} / GL(n, \mathbb{F}_q),$$

for which Freed-Quinn (1993) gives:

$$\begin{aligned} \#\{\mathcal{M}_B(GL(n, \mathbb{F}_q))\} &= \\ &= \sum_{\chi \in Irr(GL(n, \mathbb{F}_q))} \frac{|GL(n, \mathbb{F}_q)|^{2g-2}}{\chi(1)^{2g-1}} \chi(\xi_n) \end{aligned}$$

- The above count is a polynomial in q , which satisfies a certain Poincaré duality, i.e. it is a palindromic polynomial, due to the so-called Alvis-Curtis duality in the representation theory of the finite group of Lie type $GL(n, \mathbb{F}_q)$.
- This and a certain conjecture on the mixed Hodge polynomial of these character varieties leads us to conjecture that on the character variety we have the following Hard Lefschetz:

$$L^k : H^{i-k; N-k, N-k} \rightarrow H^{i+k; N+k, N+k}$$

$$L^k([\alpha]) := [\omega_I^k \wedge \alpha]$$

is an isomorphism, where N is the complex dimension of the character variety.

- This generalizes the Weak-Hard Lefschetz theorem for the moduli space of Higgs bundles.

Trivial example

When $n = 1$ then the character variety is just $(\mathbb{C}^*)^{2g}$. Reducing over the finite field \mathbb{F}_q gives $(\mathbb{F}_q^*)^{2g}$. This has $(q-1)^{2g}$ points. This is clearly a palindromic polynomial in q . Moreover the Hard Lefschetz theorem is satisfied, because the corresponding Higgs moduli space is just $T^*Jac(\Sigma)$, where we put a complex structure on Σ . Now clearly $T^*Jac(\Sigma)$ is diffeomorphic with $(\mathbb{C}^*)^{2g}$, and the Hard Lefschetz on the Kähler manifold $Jac(\Sigma)$ implies the Hard Lefschetz on the character variety.

Perhaps the many-faceted applications of the Hodge theory of these hyperkähler manifolds could lead to the:

“Quaternionic Geometry of Everything” ?