M381C Practice for the final

1. Let $f \in L^1([0,1])$. Prove that

$$\lim_{p\to 0}\left(\int_0^1 |f|^p\ dm\right)^{1/p} = \exp\int_0^1 \log|f|\ dm$$

where, by definition, $\exp(-\infty) = 0$. To simplify the problem, you may assume $\log|f| \in L^1([0,1])$. Hint: rewrite the left hand side in a form to which you can apply L'Hopital's rule.

Solution. Use L'Hopital's rule:

$$\begin{aligned}\lim_{p\to 0}\left(\int_0^1 |f|^p\ dm\right)^{1/p} &= \lim_{p\to 0}\exp\left((1/p)\log\left(\int_0^1 |f|^p\ dm\right)\right)\\ &= \exp\left(\lim_{p\to 0}\frac{\log\int_0^1 |f|^p\ dm}{p}\right)\\ &= \exp\left(\lim_{p\to 0}\frac{\int_0^1 \log(|f|)|f|^p\ dm}{\int_0^1 |f|^p\ dm}\right).\end{aligned}$$

Use the Dominated Convergence Theorem to complete the exercise.

2. Let ϕ be a differentiable Lipschitz function on $\mathbb{R}$. If f is integrable on $[a,b]$ show that the function $\Psi(t)$ defined by

$$\Psi(t) = \int_a^b \phi(tx)f(x)\ dx$$

is differentiable.

Solution.
$$\frac{\Psi(t+h)-\Psi(t)}{h} = \int_a^b \frac{(\phi((t+h)x)-\phi(tx))}{h}f(x)\ dx.$$

By assumption there is a constant $M > 0$ such that

$$\left|\frac{(\phi((t+h)x)-\phi(tx))}{h}\right| \le M|x| \le Mb.$$

So the integrand is dominated by $Mb|f(x)|$. Lebesgue's Dominated Convergence Theorem implies

$$\Psi'(t) = \int_a^b (\frac{d}{dt}\phi(tx))f(x)\ dx.$$

3. Suppose $E \subset \mathbb{R}$ is measurable and $E = E + \frac{1}{n}$ for every natural number $n \ge 1$. Show that either $m(E) = 0$ or $m(E^c) = 0$.

Hint: Fix a number N and let $F(x) = m(E \cap [N, x])$ (for $x > N$). Show that

$$F(x+\epsilon) - F(x-\epsilon) = F(y+\epsilon) - F(y-\epsilon)$$

whenever $N+\epsilon < x < y$. What does this imply about $F'(x)$? What does the Lebesgue Differentiation Theorem applied to χ_E say about F'? (This exercise can be used to show that if $G \subset \mathbb{R}$ is a proper subgroup then $m(G) = 0$).

Solution. Because $E = E + 1/n$, $F(x+\epsilon) - F(x-\epsilon) = F(x+q/n+\epsilon) - F(x+q/n-\epsilon)$ for any integer $q > 0$. Since we can choose q, n arbitrary and since F is continuous, it follows that $F\left(x+\frac{1}{n}\right) - F\left(x-\frac{1}{n}\right) = F\left(y+\frac{1}{n}\right) - F\left(y-\frac{1}{n}\right)$ whenever $N + \frac{1}{n} < x < y$. So $F'(x) = c$ for some constant c. Note that

$$F'(x) = \lim_{h\to 0} \frac{m(E \cap (x-h, x+h))}{2h}.$$

The Lebesgue Differentiation Theorem applied to χ_E says that for a.e. x,

$$\chi_E(x) = \lim_{h\to 0} \frac{m(E \cap (x-h, x+h))}{2h} = F'(x).$$

Since F' is constant a.e. we must have either $F' = 0$ a.e. in which case $m(E) = 0$ or $F' = 1$ a.e. in which case $m(E^c) = 0$.

4. Let $f \in L^1([0,1])$. What is

$$\lim_{n\to\infty} \frac{1}{n} \int_0^1 \log\left(1 + e^{nf(x)}\right)\, dx?$$

Solution. There is a constant $c > 0$ such that

$$|(1/n)\log\left(1 + e^{nf(x)}\right)| \le |f(x)| + c.$$

So Lebesgue's Dominated Convergence Theorem applies. We observe that

$$(1/n)\log\left(1 + e^{nf(x)}\right) \to f^+(x).$$

So

$$\lim_{n\to\infty} \frac{1}{n} \int_0^1 \log\left(1 + e^{nf(x)}\right)\, dx = \int_0^1 f^+\, dx.$$

5. Compute

$$\lim_{n\to\infty} \int_0^n \frac{\sin(x)}{x}\, dx.$$

Hint: Use Fubini's Theorem and the relation $1/x = \int_0^\infty e^{-xt}\, dt$ for $x > 0$.

Solution. $\pi/2$.

$$\begin{aligned}
\lim_{n\to\infty} \int_0^n \frac{\sin(x)}{x}\, dx &= \lim_{n\to\infty} \int_0^n \sin(x) \int_0^\infty e^{-xt}\, dt\, dx \\
&= \lim_{n\to\infty} \int_0^n \int_0^\infty \sin(x) e^{-xt}\, dt\, dx \\
&= \lim_{n\to\infty} \int_0^\infty \int_0^n \sin(x) e^{-xt}\, dx\, dt \\
&= \lim_{n\to\infty} \int_0^\infty \left[\frac{\cos(x) + t\sin(x)}{-1-t^2} e^{-xt}\right]_0^n dt \\
&= \lim_{n\to\infty} \int_0^\infty \frac{1}{-1-t^2} \left[(\cos(n) + t\sin(n))\, e^{-nt} - 1\right]\, dt.
\end{aligned}$$

By the dominated convergence theorem this is

$$\int_0^\infty \frac{1}{1+t^2}\,dt = \arctan(t)]_0^\infty = \pi/2.$$

6. State and prove Egoroff's Theorem.

7. Let $T : [0,1] \to [0,1]$ be an ergodic measure-preserving Borel transformation. This means that $m(T^{-1}(E)) = m(E)$ for any Borel $E \subset [0,1]$ and if $f \in L^2([0,1])$ satisfies $f \circ T = f$ then f is constant a.e. Let $\Omega = \{f \circ T - f + c : \ f \in L^\infty([0,1]), c \in \mathbb{C}\}$.

   (a) Show Ω is dense in $L^2([0,1])$. Hint: because Ω is a subspace, it suffices to show that if $v \in L^2([0,1])$ is orthogonal to every element of Ω then $v = 0$.

   (b) Show that for every $f \in L^2([0,1])$ if f_n is the function

   $$f_n = \frac{1}{n+1}\sum_{i=0}^{n} f \circ T^i$$

   then $\{f_n\}_{n=1}^\infty$ converges in $L^2([0,1])$ to the constant $\int_0^1 f\, dm$. Hint: first show this is true if $f \in \Omega$. (This is known as von Neumann's mean ergodic theorem).

8. Recall that outer measure is defined on $\mathbb{R}$ by $m^*(E) = \inf \sum_{i=1}^n m(I_i)$ where the infimum is over all collections $\{I_i\}_{i=1}^n$ of intervals that cover E and $n \in \mathbb{N} \cup \{\infty\}$. From this definition, prove that outer measure is sub-additive. That is: show $m^*(\cup_i E_i) \le \sum_i m^*(E_i)$ for any $E_1, E_2, \ldots \subset \mathbb{R}$.

9. Let μ be a finite Borel measure on $[0,1]$. Define

   $$f(x) = \mu([0,x)).$$

   Prove:

   (a) μ is absolutely continuous to Lebesgue measure if and only if f is absolutely continuous.

   (b) μ is singular to Lebesgue measure if and only if $f' = 0$ a.e.

   Hints: $\mu = \mu_{ac} + \mu_{sing}$ where μ_{ac} is absolutely continuous to Lebesgue measure and μ_{sing} is singular to Lebesgue measure. Also there is a Radon-Nikodym derivative $g = \frac{d\mu_{ac}}{dm}$. It may be easier to prove the contrapositives. For example, by showing for (b), if $f' \neq 0$ then μ is not singular to Lebesgue.

10. Find a closed subset C of $\ell^2(\mathbb{N})$ that does not have an element with smallest norm. In other words there does not exist $v \in C$ with $\|v\| = \inf\{\|w\| : \ w \in C\}$.

    Hint: If $\{e_n\}_{n\in\mathbb{N}}$ are the basis vectors and c_n's are numbers then there is an example of the form $\{c_n e_n\}_{n=1}^\infty$.

11. Let C be the set of all $x \in \ell^2(\mathbb{N})$ such that $|x_i| \le 1/i$ for all i. Show that C is compact. On the other hand, if D is the set of all $x \in \ell^2(\mathbb{N})$ with $|x_i| \le 1$ then D is noncompact. Why?

12. Define a measure on the unit sphere S^{n-1} in $\mathbb{R}^n$ as follows. For $E \subset S^{n-1}$, let $\bar{E}$ be the set of all $x \in \mathbb{R}^n$ such that $|x| \le 1$ and if $x \ne 0$ then $\frac{x}{|x|} \in E$. Define $\sigma(E) := m(\bar{E})n$. Show that this defines a measure, denoted by σ, on S^{n-1}. Moreover prove that for any measurable $X \subset \mathbb{R}^n$,

$$m(X) = \int_{S^{n-1}} \int_0^\infty \chi_X(rv) r^{n-1} \, dm(r) \, d\sigma(v).$$

Hint: first show that the formula holds whenever there is an open set $A \subset S^{n-1}$ and radii $r_1 < r_2$ such that

$$X = \{rv : \ r_1 < r < r_2, v \in A\}.$$

13. If μ and ν are signed measures on $\mathbb{R}$ then we define their convolution $\mu * \nu$ in the following way. For any set $E \subset \mathbb{R}$, let $E_2 = \{(x, y) \in \mathbb{R}^2 : \ x + y \in E\}$. Then define

$$\mu * \nu(E) := \mu \times \nu(E_2).$$

We say a measure μ is *purely atomic* if there is a countable set $C \subset \mathbb{R}$ such that $\mu(\mathbb{R} - C) = 0$. We say μ is *continuous* if $\mu(\{x\}) = 0$ for every $x \in \mathbb{R}$. Prove:

(a) if μ, ν are purely atomic then $\mu * \nu$ is purely atomic

(b) if μ is continuous then $\mu * \nu$ is continuous

(c) if $\mu << m$ then $\mu * \nu << m$.

(d) Are there measures μ, ν that are singular to Lebesgue measure such that $\mu * \nu << m$?

Hints: for (a), a purely atomic measure is a linear combination of Dirac measures (a Dirac measure is a measure supported on a single point). The convolution is bi-linear, so it suffices to show that a convolution of Dirac measures is a Dirac measure). For (b,c) you may assume μ is sigma-finite. For (d) think about measures concentrated on the Cantor set.

14. Let m denote Lebesgue measure on $[0, 1]$ and c denote counting measure on $[0, 1]$. In other words, $c(E) = |E|$, the cardinality of E. Let $\Delta = \{(x, x) : \ x \in [0, 1]\} \subset [0, 1]^2$ denote the diagonal. Compute the integrals

$$\iint \chi_\Delta(x, y) \, dm(x) \, dc(y)$$

and

$$\int \int \chi_\Delta(x, y) \, dc(x) \, dm(y).$$

Note they are not equal. Does this violate Fubini's Theorem? Does it violate Tonelli's Theorem? If not, why?

15. Identify the circle $\mathbb{T}$ with $[-\pi, \pi)$. Define the Fourier coefficents of $f \in \mathbb{L}^1(\mathbb{T})$ by

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ixn} \, dx.$$

Recall that f is **odd** if $f(x) = -f(-x)$ and it is **even** if $f(x) = f(-x)$.

(a) Show that f is odd if and only if $\hat{f}(n) = -\hat{f}(-n)$ for all n.

(b) Show that f is even if and only if $\hat{f}(n) = \hat{f}(-n)$ for all n.

16. Suppose $f \in L^1(\mathbb{T})$ and f is continuous at x. Also suppose that

$$\sum_{n=1}^{\infty} \hat{f}(n)e^{inx}$$

converges at x. Show that

$$f(x) = \sum_{n=1}^{\infty} \hat{f}(n)e^{inx}.$$

Hint: recall that the Fejér kernels F_N form an approximation to the identity.

**My apologies; I don't think this one has a solution. Instead, show that $F_N * f(x) \to x$ as $N \to \infty$. (A more difficult result is that if f is Lipschitz in a neighborhood of x then $\sum_{n=-N}^{N} \hat{f}(n)e^{inx}$ converges to $f(x)$ as $N \to \infty$. For that result, use that $D_N(x) = \frac{\sin((N+1/2)x)}{\sin(x/2)}$.