M381C Practice for the final

1. Let $f \in L^1([0,1])$. Prove that

$$\lim_{p \to 0} \left(\int_0^1 |f|^p \, dm \right)^{1/p} = \exp \int_0^1 \log |f| \, dm$$

where, by definition, $\exp(-\infty) = 0$. To simplify the problem, you may assume $\log |f| \in L^1([0,1])$. Hint: rewrite the left hand side in a form to which you can apply L'Hopital's rule.

2. Let ϕ be a differentiable Lipschitz function on $\mathbb{R}$. If f is integrable on $[a,b]$ show that the function $\Psi(t)$ defined by

$$\Psi(t) = \int_a^b \phi(tx) f(x) \, dx$$

is differentiable.

3. Suppose $E \subset \mathbb{R}$ is measurable and $E = E + \frac{1}{n}$ for every natural number $n \geq 1$. Show that either $m(E) = 0$ or $m(E^c) = 0$.

   Hint: Fix a number N and let $F(x) = m(E \cap [N, x])$ (for $x > N$). Show that

$$F(x + \epsilon) - F(x - \epsilon) = F(y + \epsilon) - F(y - \epsilon)$$

   whenever $N + \epsilon < x < y$. What does this imply about $F'(x)$? What does the Lebesgue Differentiation Theorem applied to χ_E say about F'? (This exercise can be used to show that if $G \subset \mathbb{R}$ is a proper subgroup then $m(G) = 0$).

4. Let $f \in L^1([0,1])$. What is

$$\lim_{n \to \infty} \frac{1}{n} \int_0^1 \log \left(1 + e^{nf(x)} \right) \, dx?$$

5. Compute

$$\lim_{n \to \infty} \int_0^n \frac{\sin(x)}{x} \, dx.$$

   Hint: Use Fubini's Theorem and the relation $1/x = \int_0^\infty e^{-xt} \, dt$ for $x > 0$.

6. State and prove Egoroff's Theorem.

7. Let $T : [0,1] \to [0,1]$ be an ergodic measure-preserving Borel transformation. This means that $m(T^{-1}(E)) = m(E)$ for any Borel $E \subset [0,1]$ and if $f \in L^2([0,1])$ satisfies $f \circ T = f$ then f is constant a.e. Let $\Omega = \{ f \circ T - f + c : \ f \in L^\infty([0,1]), c \in \mathbb{C} \}$.

   (a) Show Ω is dense in $L^2([0,1])$. Hint: because Ω is a subspace, it suffices to show that if $v \in L^2([0,1])$ is orthogonal to every element of Ω then $v = 0$.

(b) Show that for every $f \in L^2([0,1])$ if f_n is the function

$$f_n = \frac{1}{n+1} \sum_{i=0}^{n} f \circ T^i$$

then $\{f_n\}_{n=1}^{\infty}$ converges in $L^2([0,1])$ to the constant $\int_0^1 f \, dm$. Hint: first show this is true if $f \in \Omega$. (This is known as von Neumann's mean ergodic theorem).

8. Recall that outer measure is defined on $\mathbb{R}$ by $m^*(E) = \inf \sum_{i=1}^{n} m(I_i)$ where the infimum is over all collections $\{I_i\}_{i=1}^{n}$ of intervals that cover E and $n \in \mathbb{N} \cup \{\infty\}$. From this definition, prove that outer measure is sub-additive. That is: show $m^*(\cup_i E_i) \le \sum_i m^*(E_i)$ for any $E_1, E_2, \ldots \subset \mathbb{R}$.

9. Let μ be a finite Borel measure on $[0,1]$. Define

$$f(x) = \mu([0,x)).$$

Prove:

(a) μ is absolutely continuous to Lebesgue measure if and only if f is absolutely continuous.

(b) μ is singular to Lebesgue measure if and only if $f' = 0$ a.e.

10. Find a closed subset C of $\ell^2(\mathbb{N})$ that does not have an element with smallest norm. In other words there does not exist $v \in C$ with $\|v\| = \inf\{\|w\| : \ w \in C\}$.

11. Let C be the set of all $x \in \ell^2(\mathbb{N})$ such that $|x_i| \le 1/i$ for all i. Show that C is compact. On the other hand, if D is the set of all $x \in \ell^2(\mathbb{N})$ with $|x_i| \le 1$ then D is noncompact. Why?

12. Define a measure on the unit sphere S^{n-1} in $\mathbb{R}^n$ as follows. For $E \subset S^{n-1}$, let $\bar{E}$ be the set of all $x \in \mathbb{R}^n$ such that $|x| \le 1$ and if $x \ne 0$ then $\frac{x}{|x|} \in E$. Define $\sigma(E) := m(\bar{E})n$. Show that this defines a measure, denoted by σ, on S^{n-1}. Moreover prove that for any measurable $X \subset \mathbb{R}^n$,

$$m(X) = \int_{S^{n-1}} \int_0^{\infty} \chi_X(rv) r^{n-1} \, dm(r) \, d\sigma(v).$$

Hint: first show that the formula holds whenever there is an open set $A \subset S^{n-1}$ and radii $r_1 < r_2$ such that

$$X = \{rv : \ r_1 < r < r_2, v \in A\}.$$

13. If μ and ν are signed measures on $\mathbb{R}$ then we define their convolution $\mu * \nu$ in the following way. For any set $E \subset \mathbb{R}$, let $E_2 = \{(x,y) \in \mathbb{R}^2 : \ x + y \in E\}$. Then define

$$\mu * \nu(E) := \mu \times \nu(E_2).$$

We say a measure μ is *purely atomic* if there is a countable set $C \subset \mathbb{R}$ such that $\mu(\mathbb{R} - C) = 0$. We say μ is *continuous* if $\mu(\{x\}) = 0$ for every $x \in \mathbb{R}$. Prove:

(a) if μ, ν are purely atomic then $\mu * \nu$ is purely atomic

(b) if μ is continuous then $\mu * \nu$ is continuous

(c) if $\mu << m$ then $\mu * \nu << m$.

(d) Are there measures μ, ν that are singular to Lebesgue measure such that $\mu * \nu << m$?

14. Let m denote Lebesgue measure on $[0, 1]$ and c denote counting measure on $[0, 1]$. In other words, $c(E) = |E|$, the cardinality of E. Let $\Delta = \{(x, x) : \ x \in [0, 1]\} \subset [0, 1]^2$ denote the diagonal. Compute the integrals

$$\iint \chi_\Delta(x, y) \ dm(x) \ dc(y)$$

and

$$\int \int \chi_\Delta(x, y) \ dc(x) \ dm(y).$$

Note they are not equal. Does this violate Fubini's Theorem? Does it violate Tonelli's Theorem? If not, why?

15. Identify the circle $\mathbb{T}$ with $[-\pi, \pi)$. Define the Fourier coefficents of $f \in \mathbb{L}^1(\mathbb{T})$ by

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)e^{-ixn} \ dx.$$

Recall that f is **odd** if $f(x) = -f(-x)$ and it is **even** if $f(x) = f(-x)$.

(a) Show that f is odd if and only if $\hat{f}(n) = -\hat{f}(-n)$ for all n.

(b) Show that f is even if and only if $\hat{f}(n) = \hat{f}(-n)$ for all n.

16. Suppose $f \in L^1(\mathbb{T})$ and f is continuous at x. Also suppose that

$$\sum_{n=1}^{\infty} \hat{f}(n)e^{inx}$$

converges at x. Show that

$$f(x) = \sum_{n=1}^{\infty} \hat{f}(n)e^{inx}.$$

Hint: recall that the Fejér kernels F_N form an approximation to the identity.