**Name**________________________ **M381C Final Exam**

**Instructions**: Do as many problems as you can in 3 hours. No notes, books, googling, etc. Complete solutions (except for minor flaws) to 4 problems will be considered a good performance.

**Please place your final under my office door (RLM 9.156) by Thurs Dec 11 3pm!**

1. Suppose $f$ is absolutely continuous on $[0,1]$. Recall that $V(x) := V[f;0,x]$ is the total variation of $f$ on $[0,x]$. Show that $V(x)$ is also absolutely continuous.

   **Solution.** Let $\epsilon > 0$. Because $f$ is absolutely continuous there exists $\delta > 0$ such that if $I_1, I_2, \ldots$ is any collection of pairwise nonoverlapping intervals $I_i = [a_i, b_i]$ with $\sum_i m(I_i) < \delta$ then $\sum_i |f(b_i) - f(a_i)| < \epsilon$.

   We claim that it is also true that $\sum_i |V(b_i) - V(a_i)| < 2\epsilon$. Recall that $V(b_i) - V(a_i) = |V(b_i) - V(a_i)|$ is the total variation of $f$ in $[a_i, b_i]$. So

   $$V(b_i) - V(a_i) = \sup\{\sum_{j=1}^{n} |f(x_{i,j+1}) - f(x_{i,j})| : \ a_i = x_{i,1} < x_{i,2} < \ldots < x_{i,n} = b_i\}.$$

   So for every $i$ there exist pairwise nonoverlapping intervals $I_{ij} \subset I_i$ with $V(b_i) - V(a) \le \epsilon/2^i + \sum_j |f(d_{i,j}) - f(c_{i,j})|$ where $I_{ij} = [c_{ij}, d_{ij}]$. Therefore,

   $$\sum_i |V(b_i) - V(a_i)| \le \epsilon + \sum_{ij} |f(d_{i,j}) - f(c_{i,j})| \le 2\epsilon$$

   where the last inequality occurs because of the choice of $\delta$ and the fact that $\sum_{ij} m(I_{ij}) < \delta$.

   **Alternative solution**. Because $f$ is absolutely continuous, $f(b) - f(a) = \int_a^b f'(x)\,dx$ for any $a < b$. So $V(x) \le \int_0^x |f'(t)|\,dt$. Absolute continuity of the Lebesgue integral as a set function now implies absolutely continuous of $V$ (since $f' \in L^1$).

2. A subset $\Phi \subset L^1([0,1])$ is *uniformly integrable* if for every $\epsilon > 0$ there exists a $\delta > 0$ such that

   $$\int_E |f|\,dm < \epsilon$$

   whenever $f \in \Phi$ and $m(E) < \delta$. Suppose that $\{f_n\}_{n=1}^\infty \subset \Phi$, $f \in L^1[0,1]$ and $f_n \to f$ pointwise a.e. and $|f(x)| < \infty$ for a.e. $x$. Prove that

   $$\lim_{n\to\infty} \int_0^1 |f_n - f|\,dm = 0.$$

   **Solution**. Let $\epsilon > 0$. Let $\delta > 0$ be as in the statement. Because the integral is absolutely continuous we may assume that $\delta > 0$ is small enough so that if $E \subset [0,1]$ is any set with $m(E) < \delta$ then $\int_E |f|\,dm < \epsilon$. By Egorov's Theorem, there exists a set $X \subset [0,1]$ such that $m(X) > 1 - \delta$ and $f_n \to f$ uniformly on $X$. It follows that

   $$\limsup_{n\to\infty} \int_0^1 |f_n - f|\,dm \le \limsup_{n\to\infty} \int_X |f_n - f|\,dm + \int_{X^c} |f_n - f|\,dm$$
   $$\le \limsup_{n\to\infty} \int_{X^c} |f_n| + |f|\,dm \le 2\epsilon.$$

   Since $\epsilon > 0$ is arbitrary, this implies the statement.

3. Let $\{\mu_n\}_{n=1}^{\infty}$ be a sequence of positive Borel measures on $\mathbb{R}^k$. Define

$$\mu(E) := \sum_{n=1}^{\infty} \mu_n(E).$$

Observe that $\mu$ is a Borel measure.

(a) If every $\mu_n$ is absolutely continuous to Lebesgue measure, is $\mu$ necessarily absolutely continuous to Lebesgue measure?

(b) If every $\mu_n$ is singular to Lebesgue measure, is $\mu$ necessarily singular to Lebesgue measure?

**Solution**. (a) Let $X \subset \mathbb{R}^k$ be a set with Lebesgue measure zero. Since $\mu_n(X) = 0$ for all $n$, we must have $\mu(X) = 0$. So $\mu$ is absolutely continuous to Lebesgue.

(b) Assume each $\mu_n$ is singular to Lebesgue measure. Let $E_n \subset \mathbb{R}^k$ be a subset with $m(E_n) = 0$ and $\mu_n(E_n^c) = 0$. Observe that $\mu(\cap_n E_n^c) = 0$. Also $(\cap_n E_n^c)^c = \cup_n E_n$ has Lebesgue measure zero. So $\mu$ is singular with respect to Lebesgue.

4. If $\mu$ and $\nu$ are real-valued signed measures on $\mathbb{R}$ then we define their convolution $\mu * \nu$ in the following way. For any set $E \subset \mathbb{R}$, let $E_2 = \{(x, y) \in \mathbb{R}^2 : \ x + y \in E\}$. Then define

$$\mu * \nu(E) := \mu \times \nu(E_2).$$

The norm of a signed measure is $\|\mu\| = |\mu|(\mathbb{R})$, its total variation. Show that $\|\mu * \nu\| \leq \|\mu\|\|\nu\|$.

**Solution**. We first consider the case in which $\mu, \nu$ are positive measures. Then

$$\mu * \nu(\mathbb{R}) = \mu \times \nu(\mathbb{R}^2) = \mu(\mathbb{R})\nu(\mathbb{R}).$$

So this proves the case when $\mu, \nu$ are positive. For the general case, we may by the Hahn decomposition theorem write $\mu = \mu_1 - \mu_2$ and $\nu = \nu_1 - \nu_2$ where $\mu_1, \mu_2$ are mutually singular positive measures and $\nu_1, \nu_2$ are mutually singular positive measures. Then

$$\mu * \nu = \mu_1 * \nu_1 - \mu_2 * \nu_1 - \mu_1 * \nu_2 + \mu_2 * \nu_2.$$

So

$$|\mu * \nu| \leq \mu_1 * \nu_1 + \mu_2 * \nu_1 + \mu_1 * \nu_2 + \mu_2 * \nu_2.$$

Because $|\mu| = \mu_1 + \mu_2$ and $|\nu| = \nu_1 + \nu_2$ we have shown $|\mu * \nu| \leq |\mu| * |\nu|$. Thus $|\mu * \nu|(\mathbb{R}) \leq |\mu| * |\nu|(\mathbb{R}) = |\mu|(\mathbb{R})|\nu|(\mathbb{R})$. This implies the statement.

5. Let $f$ be a left-continuous nondecreasing nonnegative function on $[0, 1]$. Left-continuous means that $\lim_{h \searrow 0} f(x - h) = f(x)$ for any $x$. Show that there is a Borel measure $\mu$ on $[0, 1)$ such that

$$\mu([0, x)) = f(x) - f(0)$$

for any $x \in [0, 1]$. Hint: apply Carathéodory's Theorem. (We discussed this in class not long ago. However, I'd like you to show the details; for example, by justifying carefully why you can use Carathéodory's Theorem).

**Solution**. Let $\mathcal{A}$ be the collection of all finite unions of intervals of the form $[a,b)$. Observe that $\mathcal{A}$ is an algebra (it is closed under finite unions, finite intersections and complementation). Define $\mu$ on $\mathcal{A}$ by the formula

$$\mu([a_1,b_1)\cup[a_2,b_2)\cup\cdots\cup[a_n,b_m)) = \sum_i f(b_i)-f(a_i)$$

where the intervals above are disjoint. We claim that $\mu$ is countably additive in the sense that if $E_1,E_2,\ldots\in\mathcal{A}$ and $\cup_i E_i\in\mathcal{A}$ then $\mu(\cup_i E_i)=\sum_i\mu(E_i)$. By finite additivity, it suffices to prove this in the special case in which $\cup_i E_i=[a,b)$ for some $a<b$ and $E_i=[a_i,b_i)$ for some $a_i<b_i$.

We say that a number $L$ is a nontrivial limit point of the $b_i$'s if there is a sequence $\{i_j\}_{j=1}^{\infty}$ that is not eventually constant such that $L=\lim_{j\to\infty}b_{ij}$. Observe that the nontrivial limit points of the $b_i$'s are discrete. This is because if $L$ is such a limit point then there must exist $i$ such that $a_i=L$ (there does not have to exist $i$ such that $b_i=L$). So we may let $\{c_i\}_{i=1}^{\infty}$ denote the nontrivial limit points with $c_1<c_2<\ldots$. Also note that since $\{E_i\}$ is an infinite collection, nontrivial limit points exist, but there might only be finitely many. To keep the notation simple, I will assume there are infinitely many limit points, although the proof works in the finite case too.

We reindex the $E_i$'s as $E_{ij}$'s (with $0\le i<\infty, 1\le j<\infty$) satisfying $E_{ij}=[a_{ij},b_{ij})\subset[c_i,c_{i+1})$, $b_{ij}=a_{i,j+1}$, $b_{ij}\to c_i$ as $j\to\infty$ and $a_{i1}=c_{i-1}$ where, for simplicity, $c_0=a$. Then

$$\sum_{i=0}^{\infty}\sum_{j=1}^{\infty}\mu(E_{ij})=\sum_{i=0}^{\infty}\sum_{j=1}^{\infty}f(b_{ij})-f(a_{ij})=$$

$$=\sum_{i=0}^{\infty}\lim_{j\to\infty}f(b_{ij})-f(a_{i1})=\sum_{i=0}^{\infty}\lim_{j\to\infty}f(b_{ij})-f(c_{i-1})=$$

$$=\sum_{i=0}^{\infty}f(c_i)-f(c_{i-1})=\lim_{i\to\infty}f(c_i)-f(a)=f(b)-f(a)=\mu(\cup_{ij}E_{ij}).$$

We have used throughout that $f$ is left-continuous. So $\mu$ is countably additive. By Caratheodory's extension theorem, $\mu$ extends to a measure on the sigma-algebra generated by $\mathcal{A}$. That sigma-algebra contains all Borel sets (because we can obtain any open interval as a countable intersection of sets of the form $[a,b)$). So $\mu$ is a Borel measure.

**Alternative solution**. In a homework exercise, you showed that, to prove $\mu$ is countably additive, it is enough to prove that it satisfies the following continuity property: if $A_i\in\mathcal{A}$ are nested $A_1\supset A_2\supset\cdots$ and $\cap_i A_i=\emptyset$ then $\lim_i\mu(A_i)=0$. Taking complements, it is enough to prove that if $B_1\subset B_2\subset\cdots\in\mathcal{A}$ and $\cup_i B_i=[0,1)$ then $\lim_{i\to\infty}\mu(B_i)=f(1)-f(0)$. Let $b_i$ be the largest number such that $[0,b_i)\subset B_i$. Then $b_i\to 1$ as $i\to\infty$. So $\mu(B_i)\ge f(b_i)-f(0)\to f(1)-f(0)$ by left-continuity.

**Second alternative solution**. Let $G(x)$ be the infimum over all numbers $y$ such that $f(y)-f(0)\ge x$. Then $G$ is a Borel map from $[0,f(1)-f(0))$ to $[0,1)$. Observe that $\mu(E)=m(G^{-1}(E))$ for any Borel $E\subset[0,1)$.

6. Let $H_0$ be the linear span of the functions $e_n(x)=e^{inx}$ $(n\in\mathbb{Z})$ viewed as a subspace of $H=L^2([0,2\pi))$. Show that

(a) given any $g \in H_0$ the equation $f - f'' = g$ has a unique solution $f \in H_0$.

(b) The map $g \mapsto f$ is continuous as a linear map from $H_0$ to $H$, with operator norm 1.

Hint: given $f = \sum_{n\in\mathbb{Z}} a_n e_n \in H_0$ find the Fourier coefficients for $f - f''$.

**Solution**. With $f$ as above, $f - f'' = \sum_{n\in\mathbb{Z}}(1+n^2)a_n e_n \in H_0$. If $g = \sum_{n\in\mathbb{Z}} c_n e_n$ then we must have $(1+n^2)a_n = c_n$ or $a_n = \frac{c_n}{1+n^2}$. This proves existence and uniqueness. The $L^2$-norm of $g$ is $(\sum_{n\in\mathbb{Z}} |c_n|^2)^{1/2}$. The $L^2$-norm of $f$ is

$$\left(\sum_{n\in\mathbb{Z}} \left(\frac{|c_n|}{1+n^2}\right)^2\right)^{1/2} \leq \|g\|_2.$$

Since $\|f\|_2 \leq \|g\|_2$, the operator norm is at most 1. However if $g = 1$ then $f = 1$. So in this case, $\|f\|_2 = \|g\|_2$. So the operator norm is 1. This implies that the operator is continuous because bounded norm implies continuity. In fact, because it is continuous, it is uniformly continuous and therefore it can be extended to all of $L^2([0, 2\pi))$.