Homework 1 Hints

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(k)
(i)
Let K be a compact set.
compact $\Rightarrow$ closed : We will show K^c is open. Consider a point $z \in K^c$, for each point $x \in K$, we can chose two open balls U_x centered at z and V_x centered at x with radius $\frac{d(z,x)}{3}$. All such V_x forms a open cover of K, by compactness we can chose a finite subcover, $\{V_{x_k}\}$, then the intersection of the corresponding $\{U_{x_k}\}$, $\cap U_{x_k}$, is an open subset of K^c and contains z.
compact $\Rightarrow$ bounded : If otherwise, consider the open balls B_k with center x_0 and radius k.
compact $\Leftarrow$ closed and bounded : Since K is bounded, it is contained in some bounded box. Suppose for some infinite open cover of K, we cannot find a finite open cover. We split the box in to 2^n smaller box. Then there must be a box requires an infinite subcover. Keep doing the process iteratively we get a sequence of boxes $\{T_k\}$ that require infinite subcover. The intersection of $\{T_k\}$ will converge to some point p in K by closedness. And there is an open set containing p and a ball centering at p with positive radius. Then we see contradiction.
(ii)
Just prove in $\mathbb{R}^n$, closed and bounded iff sequentially compact.
(l)
Prove by contradiction. Let $d(x_n, y_n) \to 0$ with $x_n \in A$ and $y_n \in B$. Since A is compact, so $\{x_n\}$ admits a converging subsequence, use closedness of A, B to show there exists $x \in A$ and $y \in B$ with $d(x, y) = 0$.
(m)
Let $C_0 \supset C_1 \supset \ldots \supset C_k \supset \ldots$. Suppose otherwise, then $\{C_k^c\}_{k\geq 1}$ is an open cover of C_0, by compactness there is a finite subcover $\{C_{n_k}^c\}$. Let $m = \max\{n_k\}$, then C_m is empty. Contradiction.
(q)
Suppose $\{U_i\}_{i\in I}$ is an open cover of TE, then $\{f^{-1}(U_i)\}_{i\in I}$ is an open cover of E. $\{f^{-1}(U_i)\}_{i\in I}$ admits a finite subcover $\{f^{-1}(U_n)\}_n$, then $\{U_n\}_n$ is a finite subcover of TE.

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Let K be a closed subset of compact C.
Any open cover of K union with K^c is an open cover of C. It admits finite subcover. Eliminating K^c from the subcover, we get a finite subcover of K.

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Follows from **1(l)**.

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We define f on $x \in \bar{E}\backslash E$ using the limit of the $f(x_k)$ with $x_k \to x$. But we need to show the limit exists and is independent from the choice of x_k. These follow from uniform continuity. Then we can show the resulting function is continuous.

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We define refinement of a partition in the following way. If the union of the boundary points of the intervals in partition Γ' contains all the boundary points of the intervals in partition Γ and also they are on the same side of the interval. We call Γ' is a refinement of Γ. It can be shown that $L_\Gamma \le L_{\Gamma'} \le U_{\Gamma'} \le U_\Gamma$.
Suppose $|f| < M$. For given $\epsilon > 0$ and partition Γ satisfying the stated condition. Now since Γ is fixed, number of the intervals is fixed. We extend every side to the boundary point of the interval I. And we will get a refined partition of Γ, wlog we still call it Γ. Now partition Γ separates the interval I with finitely many hyperplanes. Now consider a partition $|\Gamma_1| < \sqrt{n}\delta$. Suppose the volume of a hyperplane P in $\mathbb{R}^{n-1}$ is V. If we sum up the volume of intervals in Γ_1 that intersect with P, the sum will be at most $V\delta$. Now if we refine the partition Γ_1 when there is an intersection with Γ. The change of the Darboux sum by intersecting with P will be at most $2V\delta M$. Now since the number of hyperplanes in Γ is fixed and the volume of the hyperplane in $\mathbb{R}^{n-1}$ only depends on the side lengths of the interval I and is bounded, so we can carefully choose δ so that for $|\Gamma_1| < \sqrt{n}\delta$, $L_{\Gamma_1} \ge L_\Gamma - \epsilon$ and $U_{\Gamma_1} \le U_\Gamma + \epsilon$.

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F^c is open in $\mathbb{R}^1$, so it can be written as countable union of disjoint open sets. $F^c = \cup(a_k, b_k)$, where one of a_k can be $-\infty$, one of b_k can be ∞. If $a_k = -\infty$, we set $f = f(b_k)$ on (a_k, b_k), and if $b_k = \infty$, we set $f = f(a_k)$. Otherwise we extend f on F^c by linearly interpolating f on (a_k, b_k). Then we only need to verify left continuity at $a_n \ne -\infty$ when a_n is an accumulating point of $\{a_k\}$ (right continuity at $b_n \ne \infty$ respectively). This is shown by noting f is continuous on F and interpolation does not change the extremums within any left neighborhood.