Homework 10 Hints

1
$\cup A_k = (\cup_{k=1}^N) A_k \cup (\cup_{k=N+1}^\infty A_k)$. By finitely additivity, we have $\mu(\cup A_k) = \sum_{k=1}^N \mu(A_k) + \mu(B_N)$, where $B_N = \cup_{k=N+1}^\infty A_k$. $B_N \searrow \phi$, so $\mu(B_N) \to 0$. Thus we have the countably additivity.

2
(i) $\phi = \pi_1^{-1}(\phi)$, so $\phi \in \mathcal{A}$.
(ii) If $A \in \mathcal{A}$, then for some n and B measurable, we have $A = \pi_n^{-1}(B)$. Then we have $A^c = \pi_n^{-1}(B^c)$.
(iii) If $A_1, A_2 \in \mathcal{A}$, suppose $A_i = \pi_{n_i}^{-1}(B_i)$, $i = 1, 2$ and $n_1 \leq n_2$. Then $A_1 \cup A_2 = \pi_{n_1}^{-1}(B_1 \times [0,1]^{n_2-n_1} \cup B_2)$.

3
There are various counter examples. For example, we consider $A_n = \pi_n^{-1}\{1\}$, then $\cap A_n = \{1\}^{\mathbb{N}}$ which is not in any $\mathcal{B}_n$.

4
To prove finitely additivity, we consider $A_i = \pi_{n_i}^{-1}(B_i)$, $i = 1, 2$ and $n_1 \leq n_2$, since $A_1 \cap A_2 = \phi$, so we have $B_1 \times [0,1]^{n_2-n_1} \cap B_2 = \phi$, thus $\mu_0(A_1 \cup A_2) = \pi_{n_2}^{-1}(B_1 \times [0,1]^{n_2-n_1} \cup B_2) = m_{n_2}(B_1 \times [0,1]^{n_2-n_1} \cup B_2) = m_{n_1}(B_1) + m_{n_2}(B_2) = \mu_0(A_1) + \mu_0(A_2)$.
Then we want to show if $A_1 \supset A_2 \supset \cdots$ with $\cup_n A_n = \phi$, then $\lim_{n\to\infty} \mu_0(A_n) = 0$. First we observe that if $F = \pi_n^{-1}(K)$ with K closed in $[0,1]^n$, then F is closed in $[0,1]^{\mathbb{N}}$ under the defined metric. Let $A_i = \pi_{n_i}^{-1}(B_i)$ and we can make $n_i \leq n_{i+1}$. For any $\epsilon > 0$, there is K_1 closed in $[0,1]^{n_1}$, such that $K_1 \in B_1$ and $m_{n_1}(A_1 - K_1) < \frac{1}{2}\epsilon$. Let $F_1 = \pi_{n_1}^{-1}(K_1)$. We have $\mu_0(A_1 - F_1) < \frac{1}{2}\epsilon$. In $[0,1]^{n_2}$, we can choose K_2 closed such that $K_2 \in K_1 \times [0,1]^{n_2-n_1}$ and $m_{n_2}(B_2 - K_2) < (\frac{1}{2} + \frac{1}{4})\epsilon$. Let $F_2 = \pi_{n_2}^{-1}(K_2)$, then $F_2 \subset F_1$ closed and $\mu_0(A_2 - F_2) < (\frac{1}{2} + \frac{1}{4})\epsilon$. Keep on doing the process, we have $\{F_n\}_n$ decreasing closed sets in $[0,1]^{\mathbb{N}}$, $F_n \subset A_n$ and $\mu_0(A_n - F_n) < \sum_{k=1}^n \frac{1}{2^k}\epsilon$. Since $\cap_n A_n = \phi$, so $\cap_n F_n = \phi$. But $[0,1]^{\mathbb{N}}$ is compact under the defined metric, so there exists N such that $F_N = \phi$. Then for $n \geq N$, $\mu_0(A_n) < \sum_{k=1}^N \frac{1}{2^k}\epsilon < \epsilon$. So we know $\lim_{n\to\infty} \mu_0(A_n) = 0$.

5
We just prove that $B_r(\mathbf{0})$ is measurable. $B_r(\mathbf{0}) = \{\frac{1}{2}|x_1| < r\} \cap \{\frac{1}{2}|x_1| + \frac{1}{4}|x_2| < r\} \cap \cdots$. This is a countable intersection of measurable sets in $[0,1]^{\mathbb{N}}$, so $B_r(\mathbf{0})$ is measurable.

6
$B = \{x \in [0,1]^{\mathbb{N}}, \limsup_{i\to\infty} x_i = 1\} = \cap_{k=1}^\infty \cap_{N=1}^\infty \cup_{n=N+1}^\infty \{x_n > 1 - \frac{1}{k}\}$. So it is measurable.
We note $\mu((\cup_{n=N+1}^\infty \{x_n > 1 - \frac{1}{k}\})^c) = 0$ for k, N fixed, so $\mu(\cup_{n=N+1}^\infty \{x_n > 1 - \frac{1}{k}\}) = 1$. Thus by countably additivity, $\mu(B) = 1$.