Homework 10 Hints

6.1
(a) Apply Tonelli's to $\chi_E(x,y)$.
(b) Apply Tonelli's to f, we will see all the three integrals are finite.

6.5
(a) If $\omega(y) = \infty$ for some $y > 0$, then $\int_E f = \int_0^\infty \omega(y)dy = \infty$. Otherwise $\omega(y)$ is a well-defined decreasing function on $(0,\infty)$, so it has at most countably many discontinuous points. Since $\int_E f = \int_0^\infty |\{x \in E : f(\mathbf{x} \geq y)\}|$ and $\omega(y) = |\{x \in E : f(\mathbf{x} \geq y)\}|$ a.e., following the hint on the book, we get the result.
(b) Noting that $\int_E f^p = \int_0^\infty |\{x \in E : f^p(x) > y\}|dy$, we will get the result using change of variable.

6.6
We know $f * g \in L^1$, so we can apply Fubini's. To evaluate the complex integral, we just calculate the real part and imaginary part separately. We will omit such detail here.

$$
\begin{aligned}
&\widehat{(f*g)}(x)\\
=&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t-s)g(s)ds e^{-ixt}dt\\
=&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t-s)e^{-ix(t-s)}dt g(s)e^{-ixs}ds\\
=&\hat{f}(x)\hat{g}(x)
\end{aligned}
$$

6.10
We just note that

$$
\begin{aligned}
v_n =& \int_{\sqrt{x_1^2+\cdots+x_n^2}\leq 1} 1 dx_1\cdots dx_n\\
=&2\int_0^1 \int_{\sqrt{x_1^2+\cdots+x_{n-1}^2}\leq 1-x_n^2} 1 dx_1\cdots dx_{n-1}dx_n\\
=&2\int_0^1 (1-t^2)^{\frac{n-1}{2}} v_{n-1}dt.
\end{aligned}
$$

9.2
Using Hölder's,

$$
\begin{aligned}
|(f*g)(x)| =& |\int f(t)^{p/r}g(x-t)^{q/r}\cdot f(t)^{p(1/p-1/r)}\cdot g(x-t)^{q(1/q-1/r)}dt|\\
\leq& (|f|^p * |g|^q)^{1/r}(x)\cdot\left(\int |f|^p\right)^{1/p-1/r}\cdot\left(\int |g|^q\right)^{1/q-1/r}.
\end{aligned}
$$

So

$$
\begin{aligned}
&\int |(f*g)(x)|^r dx \\
\le &\left(\int |f|^p\right)^{r/p-1} \cdot \left(\int |g|^q\right)^{r/q-1} \int (|f|^p * |g|^q)(x)dx \\
\le &\left(\int |f|^p\right)^{r/p-1} \cdot \left(\int |g|^q\right)^{r/q-1} ||f^p||_1||g^q||_1 \\
= &||f||_p^r||g||_q^r
\end{aligned}
$$

9.3
First we note $||f*K||_\infty \le ||f||_p||g||_{p'}$. Suppose $1 \le p < \infty$ wlog.
$(f*K)(x+h) - (f*K)(x) = \int (f(x+h-t) - f(x-t))K(t)dt \le ||f(x+h) - f(x)||_p||K||_{p'} \to 0$, when $h \to 0$. So we know $f*K$ is bounded and continuous.

9.4
(a) We just note taking derivative of $h(x)$ for $x > 0$ would only produce polynomial factors in front of e^{-1/x^2}.
(b) Follows from (a).
(c) Ball: $h(r - |x|)$.
Interval: $h(x_1 - a_1)h(b_1 - x_1) \cdots h(x_n - a_n)h(b_n - x_n)$.