Homework 12 Hints

8.8
Following the hint in the book,

$$\begin{aligned}\int\left|\int f(x,y)dx\right|^p dy &\le \int\left|\int f(z,y)dz\right|^{p-1}|f(x,y)|dx\,dy\\
&=\int\left|\int f(z,y)dz\right|^{p-1}|f(x,y)|dy\,dx\\
&\le\left\{\int\left|\int f(z,y)dz\right|^{(p-1)p'}dy\right\}^{1/p'}\int\left[\int|f(x,y)|^p dy\right]^{1/p}dx\\
&=\left\{\int\left|\int f(z,y)dz\right|^{p}dy\right\}^{1-1/p}\int\left[\int|f(x,y)|^p dy\right]^{1/p}dx.\end{aligned}$$

Divide both side by $\left\{\int\left|\int f(z,y)dz\right|^p dy\right\}^{1-1/p}$, we will get desired result.

9.5
Since $\overline{G_1}$ is compact and G^c is closed, so $d = d(\overline{G_1}, G^c) > 0$. Let $G_2 = \{x : d(x, \overline{G_1}) < \frac{d}{2}\}$, then G_2 is open and $\overline{G_1} \subset G_2$ and $\overline{G_2} \subset G$. Then we take $K = e^{\frac{1}{d^2/9-|x|^2}}$ and $h = \chi_{G_2} * K$.

9.6
First we note $||f * K||_\infty \le ||f||_1||K||_\infty$.
$|f*K(x+h) - f*K(x)| \le \int |K(x+h-t) - K(x-t)| \cdot |f(t)|dt$. Given $\epsilon > 0$, there is $\delta > 0$, such that if $h < \delta$, $|K(x+h-t)-K(x-t)| < \epsilon$, so $|f*K(x+h)-f*K(x)| < ||f||_1\epsilon$. This shows uniform continuity and ensures boundedness.

9.7
$f(x,y) = \int_{-\infty}^{\infty} f(t)P_y(x-t)dt$, where $P(x,y) = P_y(x) = \frac{1}{\pi}\frac{y}{y^2+x^2}$.
$\frac{f(x,y)-f(x+h,y)}{h} = \int f(t)\frac{\partial P}{\partial x}(x-t+h', y)dt$, where h' depend on x, t and $0 < h' < h$. We note $\frac{\partial P}{\partial x}(x,y) = -\frac{2xy}{(y^2+x^2)^2}$, the function is uniformly continuous and uniformly bounded as well as integrable once $y > 0$ fixed. $\int |f(t)|\left|\frac{\partial P}{\partial x}(x-t+h',y) - \frac{\partial P}{\partial x}(x-t,y)\right| dt \le \int |f_k(t)|\left|\frac{\partial P}{\partial x}(x-t+h',y) - \frac{\partial P}{\partial x}(x-t,y)\right| dt + \int |f-f_k(t)|\left|\frac{\partial P}{\partial x}(x-t+h',y) - \frac{\partial P}{\partial x}(x-t,y)\right|(1-\chi_{[-k,k]})dt$, where $f_k = f\chi_{[-k,k]}$. The first term will converge to 0 by dominated convergence. In the second term, $\left|\frac{\partial P}{\partial x}(x-t+h',y) - \frac{\partial P}{\partial x}(x-t,y)\right|(1-\chi_{[-k,k]})$ can be bounded by $|h'|$ multiplies a term which is in both L^1 and L^∞ for t, k large enough. So we can show $\frac{\partial f}{\partial x} = \int f(t)\frac{\partial P}{\partial x}(x-t,y)dt$. By similar argument we can show the desired result.
For a complete proof, see the attached pages.

9.10

The proof is similar to **Theorem (9.16)** in the book.

10.16

Since $||Tf||_p = \sup_g \frac{\int (Tf)gdx}{||g||_{p'}} \leq M||f||_p$. So $\sup_f \sup_g \frac{\int (Tf)gdx}{||g||_{p'}||f||_p} \leq M$. Since $K > 0$, we just need to check the case where $f, g > 0$. We note $\int (Tf)gdx = \int\int f(y)K(x-y)dyg(x)dx = \int (T\tilde{g})(-y)f(y)dy = \int (T\tilde{g})(y)\tilde{f}(y)dy$, where $\tilde{g}(x) = g(-x)$ and $\tilde{f}(y) = f(-y)$. So $||T\tilde{g}||_{p'} = \sup_{\tilde{f}} \frac{\int (T\tilde{g})(y)\tilde{f}(y)dy}{||\tilde{f}||} \leq M||\tilde{g}||_{p'}$. Thus we get the desired result.

10.26

(i) When $-\eta + p - 1 > 0$, $(\int_0^x f(t)dt)^p = (\int_0^x f(t)t^{\frac{\eta}{p}}t^{-\frac{\eta}{p}}dt)^p \leq (\int_0^x f^p(t)t^\eta dt)(\int_0^x (t^{-\frac{\eta}{p}})^{p'}dt)^{p/p'} = (\frac{p-1}{-\eta+p-1})^{p-1}x^{-\eta+p-1}(\int_0^x f^p(t)t^\eta dt) = cx^{-\eta+p-1}(\int_0^x f^p(t)t^\eta dt)$. So we have

$$\begin{aligned}
&\int_0^\infty (\int_0^x f(t)dt)^p x^\alpha dx \\
\leq & c\int_0^\infty x^{p-\eta-1}\int_0^x f^p(t)t^\eta dt x^\alpha dx \\
= & c\int_0^\infty \int_t^\infty x^{p-\eta-1+\alpha}dx f^p(t)t^\eta dt \\
= & -\frac{c}{p-\eta+\alpha}\int_0^\infty f^p(t)t^{\alpha+p}dt.
\end{aligned}$$

The last equality holds as long as $\alpha + p - \eta < 0$, so we should have $\alpha + p < \eta < p - 1$, which is possible for $\alpha < -1$.

(ii) Similar to (i).

($\sim$ Wheeden-Zygmund 9.7) Let $K_y = \frac{\pi^{-1}y}{x^2+y^2}$ for $x \in \mathbb{R}$ and $y > 0$. If $f_0 \in L^p(\mathbb{R})$ with $1 \le p \le \infty$, show that the *Poisson integral* $f(x,y) = (f_0 * K_y)(x)$ satisfies Laplace's equation $\Delta f = 0$ in the upper half plane $y > 0$.

We prove first that $f \in C^\infty(\{y > 0\})$. Let $P(x,y) = K_y(x)$.

Fix $x \in \mathbb{R}$ and $y > 0$. We prove that f is smooth first for $f_0 \in C_c^\infty(\mathbb{R})$ and then argue by approximation. We will also show the following formula to compute the derivatives, for a, b a given pair of non negative integers,

$$\frac{\partial^{a+b} f}{(\partial x)^a (\partial y)^b}(x,y) = \int_{\mathbb{R}} f_0(t) \frac{\partial^{a+b} P}{(\partial x)^a (\partial y)^b}(x-t,y)dt.$$

The results follows from Theorem 9.3 if $b = 0$. What we need to show is the same formula holds for any b and $a = 0$, then applying Theorem 9.3 we can recover it when $f_0 \in C_c^\infty$.

Let's proceed by induction assuming that

$$\frac{\partial^b f}{(\partial y)^b}(x,y) = \int_{\mathbb{R}} f_0(t) \frac{\partial^b P}{(\partial y)^b}(x-t,y)dt.$$

The important observation is that for $x \in \mathbb{R}$ and $y > 0$ fixed the function $\phi(t,h) : \mathbb{R} \times [-y/2, y/2] \to \mathbb{R}$ defined as

$$\phi_{x,y}(t,h) = \frac{\partial^{b+2} P}{(\partial y)^{b+2}}(x-t, y+h),$$

is globally bounded (notice also that $\|\phi_{x,y}\|_\infty$ is independent of x) and allows us to control the difference quotient (for $h \neq 0$),

$$\left| \frac{\frac{\partial^b P}{(\partial y)^b}(x-t,y+h) - \frac{\partial^b P}{(\partial y)^b}(x-t,y)}{h} - \frac{\partial^{b+1} P}{(\partial y)^{b+1}}(x-t,y) \right| \le h \left\|\phi_{x,y}\right\|_\infty.$$

It gives us the uniform convergence as h goes to zero of the following integral,

$$\left| \frac{\frac{\partial^b f}{(\partial y)^b}(x,y+h) - \frac{\partial^b f}{(\partial y)^b}(x,y)}{h} - \int_{\mathbb{R}} f_0(t) \frac{\partial^{b+1} P}{(\partial y)^{b+1}}(x-t,y)dt \right| \le$$

$$\int_{\operatorname{supp}(f_0)} \left| f_0(t) \left(\frac{\frac{\partial^b P}{(\partial y)^b}(x-t,y+h) - \frac{\partial^b P}{(\partial y)^b}(x-t,y)}{h} - \frac{\partial^{b+1} P}{(\partial y)^{b+1}}(x-t,y) \right) \right| dt$$

Therefore f has $(b+1)$ derivatives and it coincide with the proposed formula.

Now we want to show an estimate for f to be able to complete the proof for f_0 merely in L^p. We want to show that for $f_0 \in C_c^\infty(\mathbb{R})$ and any compact set $K \subseteq \{y > 0\}$ we have that

$$\left\| \frac{\partial^{a+b} f}{(\partial x)^a (\partial y)^b} \right\|_{L^\infty(K)} \le C \|f_0\|_p$$

where C depends on a, b and the (positive) distance from K to $\{y = 0\}$. By the already shown formula we have by Hölder's inequality that,

$$\left|\frac{\partial^{a+b} f}{(\partial x)^a (\partial y)^b}(x, y)\right| \leq \|f_0\|_p \left\|\frac{\partial^{a+b} P}{(\partial x)^a (\partial y)^b}(x - \cdot, y)\right\|_{p/(p-1)}.$$

But it can be shown by induction on the number of derivatives taken that the function of t $\frac{\partial^{a+b} P}{(\partial x)^a (\partial y)^b}(x-t, y)$ is integrable in $L^{p/(p-1)}$ and that its norm depends only on a, b and the distance y from (x, y) to the line $\{y = 0\}$.

Finally let $\{f_0^k\}_{k=1}^{\infty} \subseteq C_c^{\infty}(\mathbb{R})$ such that $f_0^k \to f_0$ in L^p. They generate the sequence in the upper space $\{f^k\}_{k=1}^{\infty}$ given by

$$f^k(x, y) = \int_{\mathbb{R}} f_0^k(t) P(x - t, y) dt.$$

For any pair a, b of non negative integers and any pair of indexes $k, l \geq 1$ we have that

$$\left\|\frac{\partial^{a+b} f^k}{(\partial x)^a (\partial y)^b} - \frac{\partial^{a+b} f^l}{(\partial x)^a (\partial y)^b}\right\|_{L^{\infty}(K)} \leq C \|f_0^k - f_0^l\|_p.$$

Therefore the sequence of derivatives is Cauchy and converges uniformly in compact sets to some continuous function $g_{a,b}$. It is now a known property that if a sequence of smooth functions converge uniformly over compact sets and their derivatives also converge uniformly over compact sets then the limit function is differentiable and its derivatives are the limits of the derivatives of the sequence. Moreover the same estimates we already proved remains true,

$$\left\|\frac{\partial^{a+b} f}{(\partial x)^a (\partial y)^b}\right\|_{L^{\infty}(K)} \leq C \|f_0\|_p \text{ for every } f_0 \in L^p.$$

The fact that f is harmonic follows now because P is already harmonic outside the origin (it is the imaginary part of the analytic function $z = x + iy \to 1/z$),

$$\Delta f(x, y) = \int f_0(t) \Delta P(x - t, y) dt = 0.$$