Homework 13 Hints

1

(b)⇔(c): This is easily seen using the fact μ is a probability measure.

(a)⇒(b): Let $f_k(x) = \max\{0, 1 - k \cdot d(x, F)\}$. So we gave a sequence of functions $f_k \in C_c(X)$ and $f_k \searrow \chi_F$. Then $\limsup_{n\to\infty} \mu_n(F) \leq \limsup_{n\to\infty} \int f_k d\mu_n = \int f_k d\mu \to \mu(F)$.

(b),(c)⇒(a): Suppose $0 \leq f \leq 1$. $\int f d\mu = \int_0^1 \mu(f \geq x)dx \geq \int_0^1 \limsup_{n\to\infty} \mu_n(f \geq x)dx \geq \limsup_{n\to\infty} \int_0^1 \mu_n(f \geq x)dx = \limsup_{n\to\infty} \int f d\mu_n$.

Thus we have proved $\int f d\mu \geq \limsup_{n\to\infty} \int f d\mu_n$. To prove $\liminf_{n\to\infty} \int f d\mu_n \geq \int f d\mu$ we notice that $1-f \in C_c(X)$. Or we can follow **6.5** in homework 10, using the fact that $\int f d\mu = \int_0^1 \mu(f > x)dx$.

2

Let $\mathcal{U} = \{F : F \text{ closed and } \mu(X\backslash F) = 0\}$, which essentially means that $\{X\backslash F\}_{F\in\mathcal{U}}$ are all the open sets with measure zero. $\cup_{F\in\mathcal{U}}(X\backslash F)$ is an open set. By compact separability, we know $\cup_{F\in\mathcal{U}}(X\backslash F) = \cup_{k=1}^{\infty}(X\backslash F_k)$ with $F_k \in \mathcal{U}$. Then $\mu(\cup_{F\in\mathcal{U}}(X\backslash F)) = \mu(\cup_{k=1}^{\infty}(X\backslash F_k)) \leq \sum_{k=1}^{\infty} \mu(X\backslash F_k) = 0$. And $F = \cap_{k=1}^{\infty} F_k$ is the set we are looking for. To show it is smallest, we just note that if $F' \subset F$ has the same property, then $\mu(X\backslash F') = \mu(X\backslash F) = 0$, so $X\backslash F' \subset X\backslash F$ which implies $F \subset F'$, so $F = F'$. And uniqueness follows similarly by taking the intersection.

3

Let f be the extension of Riemann-Lebesgue function on $\mathbb{R}$ such that $f(x) = 0$ for $x < 0$ and $f(x) = 1$ for $x > 1$. For interval $(a, b]$ define $\mu((a, b]) = f(b) - f(a)$. Since f is monotone and continuous, we know it is a Lebesgue-Stieltjes measure and $\mu((a, b)) = f(b) - f(a)$.

To see its support is the Cantor set C, we first notice that $\mu(C^c) = 0$. For F closed subset of C, F^c is open. If $F \neq C$, then $F^c \cap C \neq \phi$. Noticing the end points of the closed interval kept in the construction procedure is dense in C, we can thus show $\mu(F^c) \neq 0$.

4

Let $\mathbb{Q} = \{r_n\}_n$, let $\mu(B) = \sum_{n=1}^{\infty} \frac{1}{2^n} \mathbf{1}_{r_n \in B}$ for B Borel measurable.