# Homework 14 Hints

**1**

Follow the hint.

**2**

First we note that the span of $\{e_n\}$ is dense in $C([0,1])$. Then we can approximate each $e_n$ by its Taylor series expansion at $x = 0$ which is a uniform approximation on $[0,1]$. Thus we can show polynomial is dense in $C([0,1])$.

**3**

$\Rightarrow$:

For any $k \geq 0$, $f^{(k)} \in C(\mathbb{T})$. $\hat{f}^{(k)}(n) = (in)^k \hat{f}(n)$. But $\sum_{n\in Z} |\hat{f}^{(k)}(n)| < \infty$, so $|\hat{f}(n)| \leq C_k |n|^{-k}$.

$\Leftarrow$:

We can show for any $k \geq 0$, $\sum_{n\in Z} |\hat{f}(n)(in)^k| < \infty$. For $k = 0$, this implies that $f = g$ a.e. where $g = \sum \hat{f}(n)e^{inx}$ continuous and the convergence is absolute pointwise and uniform.

$$\begin{aligned}\frac{g(y)-g(x)}{y-x} &= \frac{1}{y-x}\left(\sum_{n\in Z}\hat{f}(n)e^{inx} - \sum_{n\in Z}\hat{f}(n)e^{iny}\right)\\ &= \sum_{n\in Z}\hat{f}(n)\frac{e^{iny}-e^{inx}}{y-x} = \sum_{n\in Z}\hat{f}(n)(in)e^{in\theta_n},\end{aligned}$$

where $\theta_n$ lies between $x, y$. This converges absolutely pointwise and uniformly to $\sum_{n\in Z}\hat{f}(n)(in)e^{inx}$ when $y \to x$. So we know $g' = \sum_{n\in Z}\hat{f}(n)(in)e^{inx}$ is a continuous function, so $g \in C^1(\mathbb{T})$. Then we can prove by induction $g \in C^\infty(\mathbb{T})$.