Homework 2 Hints

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Perfect set: We note that at each stage, the end points of the closed intervals are kept. Then we note that the points in the set, they are either **(a)** an end point of a closed interval that is kept at some stage or **(b)** falls in a closed interval that is kept at every stage. For case **(a)**, we note that ones the point is an end point at some stage it will always be an end point of a closed interval in the following stage, and sequence consisted of the other end point is a converging sequence to the point. For case **(b)**, we note the left end point of the small intervals is an converging sequence to the point.
Measure zero: We note the set is a close set and the complement is of measure 1.

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Perfect set and measure : Similar to **4**.
Contains no intervals: Just need to show every point is not a inner point. For case **(a)** it follows naturally. For case **(b)**, we note any neighborhood of the point x will contain a closed interval at k_x stage and then some points within the interval will be eliminated from the set in the next stage.

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We note that $|\cup_{k=n}^{\infty} E_k|_e \leq \sum_{k=n}^{\infty} |E_k|_e$ which converges to 0. Now since $\limsup E_k = \cap_{n=1}^{\infty} \cup_{k=n}^{\infty} E_k$, so $|\limsup E_k|_e \leq |\cup_{k=n}^{\infty} E_k|_e$ for any n. So $|\limsup E_k|_e = 0$. And then we note $\liminf E_k \subset \limsup E_k$.

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Measurability: we can first show that the Cartesian product of two G_δ sets is still a G_δ set. Then we use the fact a set E is measurable iff $E = H - Z$ where H is G_δ and Z of measure 0. Let $E_1 = H_1 - Z_1$ and $E_2 = H_2 - Z_2$. Then $E_1 \times E_2 = H_1 \times H_2 - (H_1 \times Z_2 \cup Z_1 \times H_2)$. Now we want to show that Cartesian product of a G_δ set and a zero measure set is still of zero measure. It suffices to show that $\mathbb{R} \times Z$ is of measure zero when $|Z| = 0$. First we need to show that $Z \times [0, 1)$ is of measure zero and this can be seen by approximating Z with open sets. Then since $\mathbb{R} = \cup[k, k + 1)$, so using the countable additivity of the measure we can show $|\mathbb{R} \times Z| = 0$.
$|E_1 \times E_2| = |E_1||E_2|$: now it suffices to show the result for G_δ sets H_1, H_2. And we only need to show for $|H_1| < \infty$ and $|H_2| < \infty$ since for unbounded set H we have $H = \cup(H \cap [k, k+1))$ and countably additivity ensures the extension to unbounded case. For bounded G_δ sets, we approximate with open sets.

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(i): Note for any $F \subset E \subset G$, we have $|F| \leq |G|$.
(ii): Enough to show that the equivalence to for any $\epsilon > 0$, there is $F \subset E$ closed such that $|E - F|_e < \epsilon$. Then note $|E|_e = |E \cap F|_e + |E - F|_e$ for $F \subset E$ closed.

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The Cantor-Lebesgue functions maps Cantor set to $[0, 1]$. Select a nonmeasurable subset of $[0, 1]$, consider the intersection of the preimage and the Cantor set.

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We will follow the hint on the book. We already see in the book(page 46) that we can find a non-measurable set $E \subset [0, 1]$ such that the rational translations of E are disjoint. The translations of E by rational r, E_r are disjoint subsets of $[-1, 2]$. Then note $\cup E_r \subset [-1, 2]$ while $|E_r|_e = |E|_e > 0$.

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Following the hint on the book, the proof will be in the following steps.
(a) Show that $|TE| = |E|$ for interval $[s2^{-k}, (s+1)2^{-k}], s = 0, 1, \ldots, 2^k - 1$.
(b) Show for open set which can be written as countable union of non-overlapping closed intervals.
(c) Show for set with small measure, the image also has small measure.
At first we briefly introduce the notion of a k-dimensional cylinder in the dyadic development. $S^{a_1,\ldots,a_k}_{p_1,\ldots,p_k} = \{x :$ the dyadic development in the position p_i is $a_i\}$. Basically, we consider each digit in the dyadic development as one dimension. And we can verify that each k-dimensional cylinder has Lebesgue measure 2^{-k} by expressing it as the union of intervals.
(a): Suppose $s = \sum_{i=0}^{i=k-1} c_i 2^i$, so $s2^{-k} = \sum_{i=1}^{k} c_{k-i} 2^{-i}$. If we denote $\beta_i = c_{k-i}$ any point $x \in [s2^{-k}, (s+1)2^{-k}]$, can be written as $x = 0.\beta_1 \ldots \beta_k \ldots$. So it actually is the cylinder $S^{\beta_1,\ldots,\beta_k}_{1,\ldots,k}$. After the transformation the image will be the points whose dyadic development have the property that in the p_i position the value is β_i which is the cylinder $S^{\beta_1,\ldots,\beta_k}_{p_1,\ldots,p_k}$.
(b): Now we just want to show that two non-overlapping cylinder after the transformation will still be non-overlapping. For two cylinders $S^{\beta_1,\ldots,\beta_n}_{p_1,\ldots,p_n}$ and $S^{\alpha_1,\ldots,\alpha_m}_{q_1,\ldots,q_m}$, the non-overlapping condition is the same as requiring that for $r \in \{p_i\}_i \cap \{q_i\}_i$, the corresponding dyadic developments in position r are not the same. And we can see that the transformation will keep this property.
(c): For set E with measure ϵ we can always find an open set $G \supset E$ such that $|G| < 2\epsilon$. But since G preserves its measure under the transformation T and $TE \subset TG$ so $|TE| < 2\epsilon$.
Finally we approximate a measurable set E using a sequence of open sets.