Homework 3 Hints

25
We will follow the hint on the book. We first construct a Cantor-type set E_1 with positive measure $1-\delta_1$ the same manner as **(5)** in HW**2**. Then the removed set is a countable union of disjoint open intervals, on each subinterval(we may include the end point since it is in E_1) we construct Cantor-type sets with ratio the removed interval length w.r.t. the original interval at kth stage $\delta_2 3^{-k}$. Then the removed set is a countable union of disjoint open intervals. We call the closed set we keep E_2. Iteratively, we define set E_k and we know the measure of E_k is $1-\delta_1\ldots\delta_k$. E_k is a increasing sequence and we define E as the limit of E_k. We can select δ_k to make $\Pi_{k=1}^{\infty}\delta_k > 0$, for example $\delta_k = \delta^{\frac{2}{2^k}}$. Then we can show E and E^c are both dense in $[0,1]$. For each interval $[a,b]$, it will contain a interval from the removed set at some stage and due to our construction a positive measure part will be in E and another positive measure part of the interval will be in E^c.

3
F measurable $\Rightarrow f, g$ measurable : Consider $F^{-1}(G\times\mathbb{R}^n)$.
f, g measurable $\Rightarrow F$ measurable : Open set on $\mathbb{R}^{2n}$ can be represented by countable union of Cartesian product of open set on $\mathbb{R}^n$ and $\mathbb{R}^n$. And we have already seen from last homework that Cartesian product of two measurable sets is measurable.

5
We follow the hint in the book. We know that Cantor-Lebesgue function F is a surjection from the Cantor Set to $[0,1]$, so if we select a non-measurable set A. The image FA under the proper inverse would be a subset of the Cantor set which is of measure zero. Then we take ϕ as the characteristic function of the image FA. We note ϕ is measurable while $(\phi(f))^{-1}(1)$ is non-measurable.
Now we need to define a measurable proper inverse f to justify our argument. We define the inverse using a limit process similar to the one when we define the Cantor set. We note Cantor Lebesgue function is an increasing function so at each stage, the inverse is well defined except at the points c such that for new end points $b_k < a_k$, $F(a_k) = F(b_k) = c$, for such case we define $f(c) = a_k$. Then at each stage we have a piecewise linear function which is measurable. And we can show that the sequence of function we defined will converge since for any x either $F(a_k) = x$ for some k and left end point a_k or it is the limit of a sequence of such points.

6
(a) $\{f+g>a\} = \cup_{r_k}(\{\infty > f > r_k\}\cap\{\infty > g > a - r_k\})\cup Z$, where Z is a subset of $\{f=\pm\infty\}\cup\{g=\pm\infty\}$.
(b)
fg measurable:
For $a>0$, $\{fg>a\} = [\cup_{r_k>0}(\{f>r_k\}\cap\{g>\frac{a}{r_k}\})]\cup[\cup_{r_k<0}(\{f<r_k\}\cap\{g<\frac{a}{r_k}\})]$.
$\{fg>0\} = (\{f>0\}\cap\{g>0\})\cup(\{f<0\}\cap\{g<0\})$.
For $a<0$, $\{fg>a\} = [\cup_{r_k>0}(\{0<f<r_k\}\cap\{0>g>\frac{a}{r_k}\})]\cup[\cup_{r_k<0}(\{0>f>r_k\}\cap\{0<g<\frac{a}{r_k}\})]\cup\{f=0\}\cup\{g=0\}\cup\{fg>0\}$.

If we define $f+g$ where it has undetermined form to be a fixed value b:
Then $\{f+g>a\} = \cup_{r_k}(\{\infty > f > r_k\} \cap \{\infty > g > a - r_k\}) \cup (\{f=\infty\} \cap \{g=\infty\}) \cup \chi_{b>a}[(\{f = -\infty\} \cap \{g=\infty\}) \cup \{f=\infty\} \cap \{g=-\infty\}]$.

8

(a) $f+g$ is usc : $\lim_{x\to x_0}\sup f(x) \le f(x_0)$ and $\lim_{x\to x_0}\sup g(x) \le g(x_0)$ so $\lim_{x\to x_0}\sup(f+g)(x) \le \lim_{x\to x_0}\sup f(x) + \lim_{x\to x_0}\sup g(x) \le f(x_0)+g(x_0)$.
$f-g$ not necessarily usc.
fg is usc when $f, g \ge 0$.
(b) $\inf_k f_k(x) \le f_k(x)$ for any k, so $\lim_{x\to x_0}\sup \inf_k f_k(x) \le \lim_{x\to x_0}\sup f_k(x) \le f_k(x_0)$ for any k.
(c) For any $\epsilon > 0$, there is $\delta > 0$ and $K > 0$ such that when $x \in B_\delta(x_0)$ and $k > K$, $|f_k(x) - f(x)| < \epsilon$.
So $\lim_{x\to x_0}\sup f(x) \le \lim_{x\to x_0}\sup f_k(x) + \epsilon \le f_k(x_0) + \epsilon \le f(x_0) + 2\epsilon$.

12

We can define the sequence of approximating simple functions as in Theorem 4.13 in the book, then the simple functions are measurable since f is continuous a.e.. Then we know as the limit f is measurable.
In $\mathbb{R}^n$, the construction is similar.