Homework 4 Hints

4.15
Proof 1:
Let $E_n = \{x \in E : \exists k, |f_k(x)| \geq n\}$, then $E_n \searrow \phi$. Since $E_0 = E$ is of finite measure, so $\lim_n |E_n| = |\lim_n E_n| = 0$, so for any $\epsilon > 0$, there is N with $|E_N| < \frac{\epsilon}{2}$, then on $E \backslash E_N$ we have $|f_k(x)| < N$ for any k, since E_N is measurable, so there is $F \subset E \backslash E_N$ with $|E_N \backslash F| < \frac{\epsilon}{2}$.
Proof 2:
Consider $f = sup_k |f_k|$, from Lusin's Theorem we know f is continuous on a closed subset F' of E with $|E \backslash F'| < \frac{\epsilon}{2}$. Since $|E| < \infty$, we can find $F \subset F'$ which is bounded and $|E \backslash F| < \epsilon$ and on F, f has a maximum.

4.18
(i)$\{f > a\} = \{f_k > a\} \cup (\{f > a\} \cap \{f_k \leq a\})$. But since $f_k \nearrow f$, $\{f > a\} \cap \{f_k \leq a\}$ converges to empty set, so $\{f_k > a\} \nearrow \{f > a\}$.
(ii)Follow the hint on the book, we need to show that $\limsup_{k\to\infty} \omega_{f_k}(a) \leq \omega_f(a - \epsilon)$ for any $\epsilon > 0$, then the continuity of ω_f will prove the result. We note that $\{f_k > a\} \subset \{f > a - \epsilon\} \cup (\{f_k > a\} \cap \{f \leq a - \epsilon\}) \subset \{f > a - \epsilon\} \cup \{|f - f_k| > \epsilon\}$. But convergence in measure ensures that $|\{|f - f_k| > \epsilon\}| \to 0$.

4.19
We define $f_n(x, y) = f(x, \frac{k}{n})$ where $\frac{k}{n} \leq y < \frac{k+1}{n}$. Then we can show f_n is continuous a.e. thus measurable. The continuity of f in y ensures that $f = \lim f_n$.

5.2
For (i) we consider $f_k = -\chi_{[k,\infty)}$ and for (ii) consider $f_k = \chi_{[k,\infty)}$.

5.4
Note that $|x^k f(x)| \leq |f(x)|$ and $x^k f(x) \to 0$ in $(0, 1)$. Then apply Dominated Convergence theorem.

5.9
Let $E_\epsilon = \{x \in E : |f_k - f| > \epsilon\}$, then $\int_{E_\epsilon} |f - f_k|^p \geq \epsilon^p m(\{|f_k - f| > \epsilon\}) \to 0$, so $m(\{|f_k - f| > \epsilon\}) \to 0$.

5.13
(i)$\int_E \sum |f_k| = \sum \int_E |f_k|$, so $\sum |f_k| < \infty$ a.e..
(ii) We can verify that $\int_{[0,1]} |x - r_k|^{-\frac{1}{2}} < 4$, then use (i).