Homework 6 Hints

7.1

Since $m = m(\{|f| > 0\}) > 0$, so there is $\alpha > 0$ such that $m(\{|f| > \alpha\}) > \frac{m}{2}$. Then there is r such that $m(\{|f| > \alpha\} \cap \{|x| < r\}) > \frac{m}{4}$ and we note that $\int_{\{|f|>\alpha\}\cap\{|x|<r\}} |f| > \frac{\alpha m}{4}$. For any x, a ball centered at x with radius $|x|+r$ will contain the set $\{|f| > \alpha\} \cap \{|x| < r\}$. So there is a cube Q_x with side length $2(|x|+r)$ containing the set. Then $f^*(x) \geq \frac{1}{|Q_x|}\frac{\alpha m}{4} = \frac{m\alpha}{2^{n+2}|x|^n}\frac{|x|^n}{(|x|+r)^n} \geq \frac{m\alpha}{2^{n+2}(r+1)^n}|x|^{-n}$.

7.2

Suppose $|\phi| < M$Since $\int \phi = 1$, so

$$
\begin{aligned}
(f * \phi_\epsilon)(x) - f(x) &= \int (f(x-y) - f(x))\phi_\epsilon(y)dy \\
&\leq \frac{1}{\epsilon^n}\int |f(x-y) - f(x)|\phi(\frac{y}{\epsilon})dy \\
&\leq \frac{c_n}{|B_\epsilon(\mathbf{0})|}\int_{B_\epsilon(\mathbf{0})} |f(x-y) - f(x)|Mdy.
\end{aligned}
$$

And we know $\frac{c_n}{|B_\epsilon(\mathbf{0})|}\int_{B_\epsilon(\mathbf{0})} |f(x-y) - f(x)|Mdy$ will converge to 0 in the Lebesgue set of f.

7.4

This is a generalization of **(3.37)** in the book and we will adopt a similar proof. Following the proof of **(3.37)**, we have $\mathcal{E}_i$ subset of E_i and interval I_i such that $|\mathcal{E}_i| > \frac{3}{4}|I_i|$. Supposing $|I_1| \geq |I_2|$, there is $q \in \mathbb{N}$ such that $q|I_2| \leq |I_1| < (q+1)|I_2|$. We divide I_1 into q non-overlapping subintervals with equal length. There must be a subinterval $\tilde{I}_1$ such that $|\mathcal{E}_1 \cap \tilde{I}_1| > \frac{3}{4}|\tilde{I}_1|$. And we have $\frac{q}{q+1}|\tilde{I}_1| < |I_2| \leq |\tilde{I}_1|$. Since $q \geq 1$ so $\frac{1}{2}|\tilde{I}_1| < |I_2| \leq |\tilde{I}_1|$. We still denote $\tilde{I}_1$ as I_1 and $\mathcal{E}_1 \cap \tilde{I}_1$ as $\mathcal{E}_1$. We align the left end point of I_1 and I_2 by a translation of I_2 with τ, then we note for $d < \frac{3}{4}|I_2| - \frac{1}{4}|I_1|$, $\mathcal{E}_1$ and $\mathcal{E}_2 + \tau + d$ must have a nonempty intersection, since $|I_1| + d < \frac{3}{4}|I_1| + \frac{3}{4}|I_2| \leq |\mathcal{E}_1| + |\mathcal{E}_2 + \tau + d|$.

2.1

$|x\sin(\frac{1}{x})| < |x| \leq 1$, so f is bounded. To show it is not of bounded variation, consider the points $\frac{1}{(\frac{1}{2}+n)\pi}$.

2.4

For any partition Γ, $S_\Gamma[f_k] \to S_\Gamma[f]$ by pointwise convergence.
Consider f_n to be the function in **2.1** but is defined to be 0 on $[0, \frac{1}{n}]$.

2.7

We will follow the hint on the book. Let $\Gamma^n = \{x_i\}_n$, then $0 \leq V[x_i - 1, x_i] - |f(x_i - 1) - f(x_i)| \leq V[a,b] - S_{\Gamma^n}$. For any y in $[a,b]$ and any sequence $y_k \to y$, we just take y and $y_k$to be in the partition

Γ^k and adjacent. We suppose $y > y_k$ wlog. Since f is continuous, $|f(y) - f(y_i)| \to 0$, from **(2.9)** in the book, $V[a,b] - S_{\Gamma^n} \to 0$, so $V[y_i, y] \to 0$. This shows the continuity of V at y. Then we note $P[a,x] = \frac{1}{2}(V[a,x] + f(x) - f(a))$ and $N[a,x] = \frac{1}{2}(V[a,x] - f(x) + f(a))$.