Homework 7 Hints

7.6
Check the derivative of x^α, $\alpha x^{\alpha-1}$, satisfies **7.29**.

7.10
Note the fact that the image of $[a_i, b_i]$ is an interval of length at most $V(b_i) - V(a_i)$. Then for a set Z with measure zero, we have for any $\epsilon > 0$, $Z \in G_\epsilon$ where $|G_\epsilon| < \epsilon$. Use the fact G_ϵ it is the countable union of non-overlapping intervals and f is absolutely continuous we can show the image of Z is of measure zero.
For any measurable set, it can written as the union of a $F_\sigma \cup Z$. Considering $[a, b]$ is bounded and the image of a compact set is compact since f is absolutely continuous, the image of a measurable subset of $[a, b]$ is still measurable.

7.11
(a) Measurability follows from the fact that the preimage of an open set is measurable.
(b) Firs we note if $f = \chi_{(a,b)}$, the result follows from **(7.29)**. Then we can show the result for f the characteristic function of an open set. For characteristic function with measure zero, we can show the result by approximating with open set $|G_\epsilon| < \epsilon$. So we can prove the result for characteristic function of measurable set. Then approximate measurable function with simple functions and apply Dominated convergence.

7.12
To prove $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$, consider $x_1 = p \ln a$ and $x_2 = q \ln b$.
To prove the generalization, use the hint on the book.

7.14
ϕ is convex $\Rightarrow$ continuous and $\phi(\frac{x_1+x_2}{2}) \leq \frac{\phi(x_1)+\phi(x_2)}{2}$ follows from **(7.40)**.
ϕ is convex $\Leftarrow$ continuous and $\phi(\frac{x_1+x_2}{2}) \leq \frac{\phi(x_1)+\phi(x_2)}{2}$
We can prove by induction that $\phi(\frac{r}{2^k}x_1 + \frac{2^k-r}{2^k}x_2) \leq \frac{r}{2^k}\phi(x_1) + \frac{2^k-r}{2^k}\phi(x_2)$.

$$\begin{aligned}
&\phi(\frac{r}{2^{k+1}}x_1 + \frac{2^{k+1}-r}{2^{k+1}}x_2) \\
=&\phi(\frac{1}{2}(\frac{r}{2^k}x_1 + \frac{2^k-r}{2^k}x_2) + \frac{1}{2}x_2) \\
\leq&\frac{1}{2}\phi(\frac{r}{2^k}x_1 + \frac{2^k-r}{2^k}x_2) + \frac{1}{2}\phi(x_2) \\
\leq&\frac{r}{2^{k+1}}\phi(x_1) + \frac{2^{k+1}-r}{2^{k+1}}\phi(x_2).
\end{aligned}$$

Then the result holds by continuity.

8.2

(a) $||f||_\infty \le \sup \int_E fg$ with $||g||_1 = 1$.

If $||f||_\infty = 0$ then $f = 0$ a.e., the result holds. If $||f||_\infty < \infty$, then for any $\epsilon > 0$, $m_\epsilon = |\{x \in E : f(x) > ||f||_\infty - \epsilon\}| > 0$, consider $g_\epsilon = \text{sign}(f)\frac{1}{m_\epsilon}$. If $||f||_\infty = \infty$, use similar argument for $f_k = \min(|f|, k)$ and then let $k \to \infty$.

(b) $||f||_1 \le \sup \int_E fg$ with $||g||_\infty = 1$.

If $||f||_1 < \infty$, consider $g = \text{sign}(f)$. If $||f||_\infty = \infty$, use similar argument for $f_k = \chi_{\{|f|<k\}}\chi_{\{|x|<k\}}f$ and then let $k \to \infty$.

(c) For $f \notin L^p(E)$, there exists a function $g \in L^{p'}(E)$ such that $fg \notin L^1(E)$.

For any $k > 0$, there is $g_k \in L^{p'}(E)$ such that $\int_E fg_k \ge k$ with $||g_k||_{p'} \le 1$. Then let $a_k = \frac{1}{k^2}$ and consider $g = \sum a_k g_k$.