Homework 8 Hints

8.4
Holder's inequality: $|\int fg| \leq ||f||_p ||g||_{p'}$. When $||f||_p = ||g||_{p'} = 1$.

$$\begin{aligned} |\int fg| &\leq \int |fg| \\ &\leq \int_E \left(\frac{|f|^p}{p} + \frac{|g|^{p'}}{p'} \right) \\ &= \int ||f||_p ||g||_{p'} \end{aligned}$$

Equality holds in the first inequality iff fg is always of the same sign. In the second one, equality holds iff $|f|^p = |g|^{p'}$ a.e.. More generally $\frac{|f|^p}{||f||_p} = \frac{|g|^{p'}}{||g||_{p'}}$.
Minkowski's inequality: $||f+g||_p \leq ||f||_p + ||g||_p$.

$$\begin{aligned} ||f+g||_p^p &= \int_E |f+g|^{p-1}|f+g| \\ &\leq \int_E |f+g|^{p-1}|f| + \int_E |f+g|^{p-1}|g| \\ &\leq ||f+g||_p^{p-1}||f||_p + ||f+g||_p^{p-1}||g||_p \end{aligned}$$

Last inequality is equality iff $|f+g|$ is multiple of $|f|(|g|)$.

8.5
$N_p[f] = \left(\frac{1}{|E|} \int_E |f|^p \right)^{1/p}$. To show if $p_1 < p_2$, then $N_{p_1}[f] \leq N_{p_2}[f]$, we consider the following:

$$\begin{aligned} \int_E |f|^{p_1} &\leq (\int_E 1)^{1-\frac{p_1}{p_2}} (\int_E (|f|^{p_1})^{p_2/p_1})^{\frac{p_1}{p_2}} \\ &= |E|^{1-\frac{p_1}{p_2}} (\int_E |f|^{p_2})^{\frac{p_1}{p_2}}. \end{aligned}$$

8.6
Generalization of Holder's inequality. If $\sum_{i=1}^k \frac{1}{p_i} = \frac{1}{r}$, where $p_i, r \geq 1$, then

$$||f_1 \cdots f_k||_r \leq ||f_1||_{p_1} \cdots ||f_k||_{p_k}.$$

When $k = 2$, we have from Holder's inequality,

$$\begin{aligned} \int |f_1 f_2|^r &\leq ||f_1^r||_{p_1/r} ||f_2^r||_{p_2/r} \\ &= ||f_1||_{p_1}^r ||f_2||_{p_2}^r. \end{aligned}$$

Then we prove by induction,

$$\begin{aligned}\int |f_1 \cdots f_{k+1}|^r \leq & ||f_1 \cdots f_k||_{\frac{1}{\sum_{i=1}^k p_i}} ||f_{k+1}||_{p_{k+1}} \\ \leq & ||f_1||_{p_1} \cdots ||f_{k+1}||_{p_{k+1}}.\end{aligned}$$

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8.11

$$||f_k g_k - fg||_p \leq ||f_k g_k - f g_k||_p + ||f g_k - fg||_p$$

$||f_k g_k - f g_k||_p \leq M||f_k - f||_p$, so the first term on RHS converges to zero. $|f g_k - fg| \leq 2M|f|$, since $p < \infty$, by dominated convergence we have the second term on RHS also converges to zero.

8.12

(a)$\big| ||f||_p - ||f_k||_p \big| \leq ||f - f_k||_p \to 0$.

(b)$|f - f_k|^p \leq 2^p(|f|^p + |f_k|^p)$. Let $g_k = 2^p(|f|^p + |f_k|^p) - |f - f_k|^p$, then $\lim_{k\to\infty} g_k = 2^{p+1}|f|^p$ a.e. So by Fatou's Lemma,

$$\int 2^{p+1}|f|^p \leq \liminf \left(\int 2^p|f|^p + \int 2^p|f_k|^p + \int |f - f_k|^p \right) \tag{1}$$

By the convergence of L^p norm, we have $\limsup \int |f - f_k|^p = 0$.

8.13

For any $\epsilon > 0$, we can select $\delta > 0$ such that if $|E| < \delta$, then $\int_E |g|^{p'} < \epsilon$. And we can find N, such that $\int_{B_N(\mathbf{0})^c} |g|^{p'} \leq \epsilon$. Then by Egorov's theorem, we can find $F \subset B_N(\mathbf{0})$ such that $|B_N(\mathbf{0}) - F| < \delta$ and K such that for $k > K$, $|f - f_k|^p < \frac{\epsilon}{|F|}$ on F. Then for $k > K$,

$$\begin{aligned}\int |f - f_k||g| = & \int_{B_N(\mathbf{0})^c} |f - f_k||g| + \int_F |f - f_k||g| + \int_{B_N(\mathbf{0}) - F} |f - f_k||g| \\ \leq & 2M\epsilon^{1/p'} + \epsilon^{1/p}||g||_{p'} + 2M\epsilon^{1/p'}.\end{aligned}$$

8.14a

Straight forward calculation.

8.15

We note $\int_0^{2\pi} \cos^2(kx)dx = \pi$. So for $f \in L^2$ the Fourier coefficient is $\sqrt{\frac{1}{\pi}} \int f \cos(kx)dx$. However the sum of square of the sequence is finite, thus $\int f \cos(kx)dx \to 0$. For $f \in L^1$, we can approximate by compactly supported continuous f_n, which have the above property. Since $\cos(kx)$ is bounded, we have $\int |\cos(kx)||f - f_n| \to 0$. So $\int f \cos(kx) \to 0$.