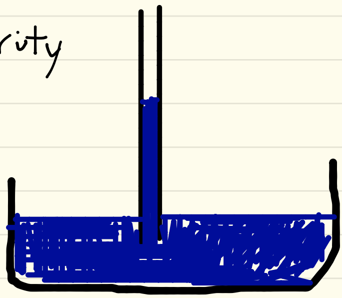
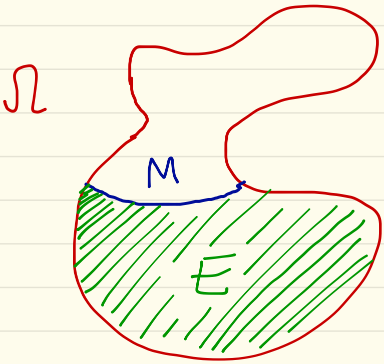
Regularity of free boundaries in anisotropic capillarity problems & the validity of Young's Law

joint work with Guido De Philippis (U. Zürich)

The capillarity tube



Capillarity Problem (Young, Laplace 1805, Gauß 1830)



Fluid in a container

$\Omega \subseteq \mathbb{R}^n$ the container

E the region occupied by the fluid

$M = \partial E \cap \Omega$
= the liquid-air interface

Gauß free energy

Potential energy
(typically $g(x) = g_0 \rho x_n$, $n=3$)

Lagrange multiplier
(volume constraint)

$$\mathcal{H}^{n-1}(M) + \int_{\partial E \cap \partial \Omega} \sigma(x)\, d\mathcal{H}^{n-1}(x) + \int_E g(x)\, dx - \lambda |E|$$

Adhesion coefficient, $|\sigma| \leq 1$

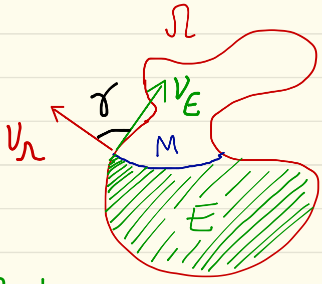
Total surface tension energy of liquid-air interface

Wetted surface $\partial E \cap \partial \Omega$

Total surface tension energy of liquid-solid interface

Euler-Lagrange eqns.

$$\mathcal{H}^{n-1}(M) + \int_{\partial E \cap \partial\Omega} \sigma(x)\, d\mathcal{H}^{n-1}(x) + \int_E g(x)\,dx - \lambda|E|$$



(1) $H_M(x) + g(x) = \lambda$ for $x \in M \cap \Omega$

(2) $\nu_E(x) \cdot \nu_\Omega(x) = \sigma(x)$ for $x \in M \cap \partial\Omega$

Rmks

(i) When $M = \{(x, u(x)) : x \in G\}$, $\Omega = G \times \mathbb{R}$, then (1) becomes

$$-\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) + g(x, u(x)) = \lambda \quad \text{on } G$$

(ii) (2) is called Young's Law. It is insensitive of potential energy & it implies that $-1 \le \sigma(x) \le 1$ $\forall x \in \partial\Omega$.

It says $\gamma(x)$ = contact angle is determined by $\sigma(x)$.

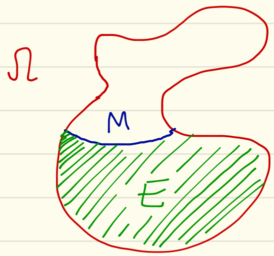
(iii) Huge literature assuming the validity of (1) & (2)

Global minimizers: for $0<m<1$ given, solve

$$\inf\left\{ \mathcal{H}^{n-1}(M) + \int_{\partial E\cap\partial\Omega} \sigma(x)\, d\mathcal{H}^{n-1}(x) + \int_E g(x)\,dx \;:\; |E| = m|\Omega| \right\}$$

Existence is obtained in the class of sets of finite perim.

$\Rightarrow$ a priori we just know that



$M = \bigcup_{h\in\mathbb{N}} K_h$, K_h compact $\subseteq M_h$

M_h C^1-hypersurface

Interior regularity $\Rightarrow$ validates

$$H_M(x) + g(x) = \lambda \qquad \text{for } x \in M\cap\Omega$$

Boundary regularity $\Rightarrow$ validates
(free)

$$\nu_E(x)\cdot\nu_\Omega(x) = \sigma(x) \qquad \text{for } x\in M\cap\partial\Omega$$

Rmk: $\sigma=0,\ g=0 \Rightarrow$ Relative isoperimetric problem in Ω
as $\mathcal{H}^{n-1}(M) = \mathcal{H}^{n-1}(\Omega\cap\partial E) = P(E;\Omega)$

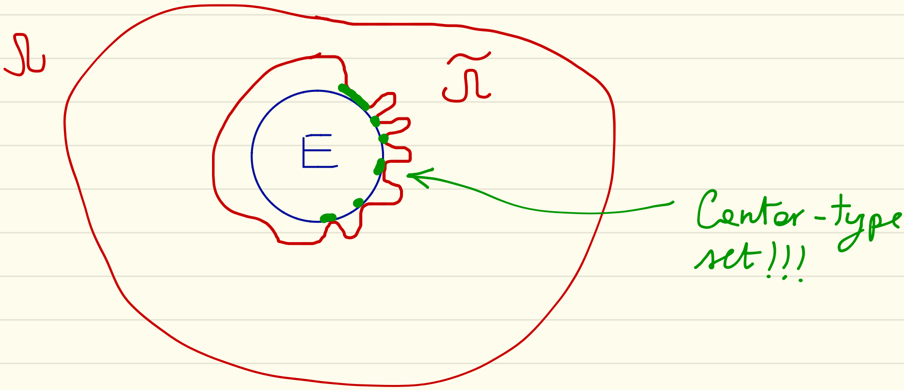
Geometric motivation to study the problem in every dimension!

Rmk.1 to expect regularity one needs $-1<\sigma<1$

For ex. consider $g=0$, $\sigma\equiv 1$, m small

then $\mathcal{H}^{n-1}(M)+\int_{\partial E\cap\partial\Omega}\sigma = \mathcal{H}^{n-1}(\partial E) = P(E)$

$\Rightarrow$ $\inf\{P(E) : |E| = m|\Omega|\}$ so that E is a ball of volume $m|\Omega|$ & E is also a global minimizer w.r.t. its own volume in any $\tilde{\Omega}$ s.t. $E\subseteq\tilde{\Omega}\subseteq\Omega$ ...



Rmk2: When $\sigma=-1$ non existence issues! For ex.

$$\inf\{\mathcal{H}^{n-1}(\partial E\cap\{x_n>0\}) - \mathcal{H}^{n-1}(\partial E\cap\{x_n=0\}) : |E|=m\} = 0$$

Interior regularity (De Giorgi; Federer, Almgren...)

"$M \cap \Omega$ is smooth as much as $g(x)$ allows it to be out of a closed set Σ_{int} of codimension at least 8"

Boundary regularity

J. Taylor (77) $n=3$, $M \cap \partial\Omega$ is a smooth curve (as much as σ, g and $\partial\Omega$ allow it to be)

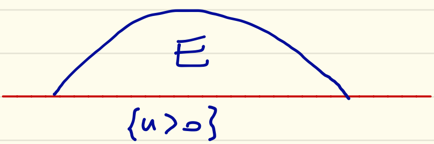
Caffarelli & Friedman (85) $2 \leq n \leq 7$ $\quad -1 < \sigma(x) < 0$

$\Omega = \{x_n > 0\} \quad g(x) = g(x_n)$

Sessile droplet problem

$E = \{(x,t) : 0 < t < u(x)\}$

for some $u : \mathbb{R}^{n-1} \to [0,\infty)$, by symmetrization.



Wetted surface $= \{u>0\}$. If u lipschitz, then

$\Rightarrow$ use reg. theory for free boundaries to conclude

Barrier argument

Griiter & Griiter-Jost (various papers)

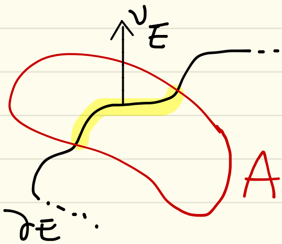
Every n, $\sigma \equiv 0$, reflection trick at boundary to use interior regularity

Anisotropic surface tension/perimeter

$\Phi = \Phi(x,\nu) : \Omega \times S^{n-1} \longrightarrow (0,\infty)$ ELLIPTIC, i.e.

$\Phi(x,\cdot)$ extended by 1-homogeneity is convex on $\mathbb{R}^n$

$$\Phi(E;A) = \int_{A\cap\partial E} \Phi(x,\nu_E(x))\, d\mathcal{H}^{n-1}(x)$$



Crystallography (Wulff problem)

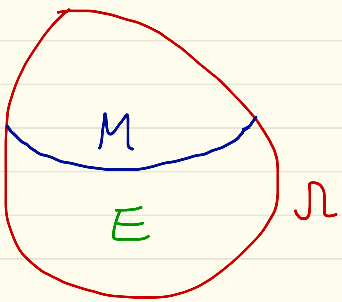
Isoperimetric pbs. in Riemannian /Finsler manifolds

Stationariety conditions for

$$\Phi(E;\Omega) + \int_{\partial E\cap\partial\Omega} \sigma + \int_E g - \lambda|E|$$

(1) $-\mathrm{div}_M\left(\nabla\Phi(x,\nu_E)\right) + \nu_E\cdot\nabla_x\Phi + g = \lambda$ on $M\cap\Omega$

(2) $\nabla\Phi(x,\nu_E)\cdot\nu_\Omega = \sigma$ on $M\cap\partial\Omega$

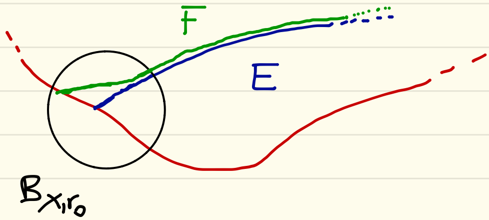


$M = \mathrm{cl}(\Omega\cap\partial E)$

Rmk. If $\Phi(x,\nu) = |\nu|$, then we are back to the capillarity prob. with Young-Laplace eqns

Local almost minimizers of $\quad \Phi(E;\Omega) + \int_{\partial E \cap \partial\Omega} \sigma = \mathcal{F}(E)$

$$\mathcal{F}(E) \le \mathcal{F}(F) + \Lambda |E \Delta F| \quad \forall F \subseteq \Omega, \; E \Delta F \subset\subset B_{x,r_0}$$



Dirichlet condition on $\partial B_{x,r_0} \cap \Omega$

Neumann condition on $B_{x,r_0} \cap \partial\Omega$

Interior regularity (Schoen-Simon, Almgren, Bombieri)

If $\Phi(x,\cdot)$ is smooth & uniformly elliptic,
then $M = cl(\Omega \cap \partial E)$ is smooth outside of a closed set Σ_{in}
with $\mathcal{H}^{n-3}(\Sigma_{in}) = 0$.

(also works with more general notions of almost-minimizer)

Boundary regularity seems open, with the ideas developed in the isotropic case not so obviously adaptable

$$\boxed{\mathcal{F}(E) = \Phi(E;\Omega) + \int_{\partial E \cap \partial\Omega} \sigma}$$

Theorem (w. G. De Philippis) IF (i) Ω open, $\partial\Omega$ smooth

(ii) $\Phi : \Omega \times S^{n-1} \to (0,\infty)$ $C^{2,1}$, unif. Lipschitz in x, unif. elliptic in ν, i.e. $\begin{cases} \lambda^{-1} \le \Phi \le \lambda \\ \nabla^2\Phi(x,\nu)[\tau,\tau] \ge \frac{|\tau|^2}{\lambda} \quad \forall \tau \in \nu^\perp \end{cases}$

(iii) $\sigma \in \mathrm{Lip}(\partial\Omega)$ with $-\Phi(x,-\nu_\Omega) < \sigma < \Phi(x,\nu_\Omega)$ on $\partial\Omega$

(iv) $E \subseteq \Omega$, $\mathcal{F}(E) \le \mathcal{F}(F) + \Lambda|E \Delta F|$ $\forall F \subset \Omega$, $E \Delta F \subset\subset B_{x,r_0}$

THEN (a) E is open, $\partial E \cap \partial\Omega$ is of finite perimeter in $\partial\Omega$

(b) if $M = \mathrm{cl}(\Omega \cap \partial E)$, then $M \cap \partial\Omega = \partial_{(\partial\Omega)}(\partial\Omega \cap \partial E)$ and there exists $\Sigma \subseteq M$ closed s.t.

(b1) $M \setminus \Sigma$ is a $C^{1,1/2}$ hypersurface with boundary

(b2) $\nabla\Phi(x,\nu_E)\cdot\nu_\Omega = \sigma$ on $(M\setminus\Sigma)\cap\partial\Omega$

(b3) $\mathcal{H}^{n-3}(\Sigma) = 0$.

Rmk. $\mathcal{H}^{n-3}(\Sigma \cap \Omega) = 0$ due to Schoen, Simon & Almgren

We prove $\mathcal{H}^{n-3}(\Sigma \cap \partial\Omega) = 0$.

Proof. **Step one**: We change Ω into $\{x_1>0\}$ by $\partial\Omega \in C^\infty$

Φ anisotropic & E is Λ-minimizing are stable under smooth diffeos

Step two: We get rid of σ: in the case, say, that $\Phi = |\nu|$ & σ is a constant (so that (iii) gives $-1<\sigma<1$)

$$\begin{aligned}
\mathcal{F}(E) &= \mathcal{H}^{n-1}(\partial E \cap \Omega) + \sigma\, \mathcal{H}^{n-1}(\partial E \cap \partial\Omega) \\
&= \mathcal{H}^{n-1}(\partial E \cap \Omega) + \int_{\partial E \cap \partial \Omega} \sigma(-e_1)\cdot \nu_\Omega \\
&= \int_{\Omega \cap \partial E} (|\nu_E| + \sigma\, e_1 \cdot \nu_E)\, d\mathcal{H}^{n-1}
\end{aligned}
\quad
\left\{
\begin{aligned}
&\Omega = \{x_1>0\} \\
&\nu_\Omega = -e_1 \\
&\operatorname{div}(\sigma e_1) = 0
\end{aligned}
\right.$$

Thus $\left\{ \begin{array}{l} E \ \Lambda\text{-minimizer} \\ \text{of } \mathcal{F} \text{ with } \Phi = |\nu| \\ \qquad -1<\sigma<1 \end{array} \right. \Rightarrow \left\{ \begin{array}{l} E \ \Lambda\text{-minimizer of} \\ \mathcal{F} \text{ with } \Phi = |\nu| + \sigma(\nu\cdot e_1) \\ \qquad \sigma = 0 \end{array} \right.$

Note that $\Phi = |\nu| + \sigma(\nu\cdot e_1)$ is still elliptic iff $\boxed{-1<\sigma<1.}$

Step three: We prove an ε-regularity criterion

Let $\displaystyle exc(E,x,r)=\inf\left\{\frac{1}{r^{n-1}}\int_{B_{x,r}\cap\Omega\cap\partial E}|\nu_E-\nu|^2\,d\mathcal{H}^{n-1} : \nu\in S^{n-1}\right\}$

There exist $\varepsilon=\varepsilon(n,\lambda)$, $C=C(n,\lambda)$, $\beta=\beta(n,\lambda)$ s.t.

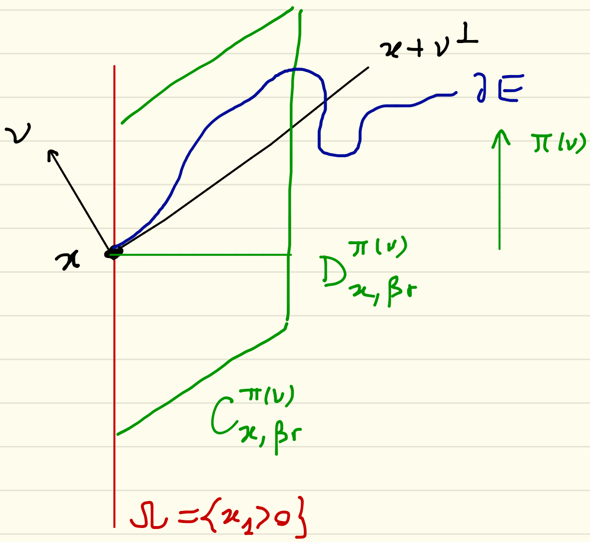
IF $exc(E,x,r)+(\Lambda+\ell)r<\varepsilon(n,\lambda)$

THEN $\exists\,\nu\in S^{n-1}$ s.t. $\nabla\Phi(x,\nu)\cdot e_1=0$ $(\Omega=\{x_1>0\},\ \nu_\Omega=-e_1)$

$$|\nu\cdot e_1|<1-1/C$$

$$\exists\, u: D^{\pi(\nu)}_{x,\beta r}\to\mathbb{R} \text{ with } \frac{|u|}{r}+|\nabla u|+\sqrt{r}\,[\nabla u]_{C^{1/2}}\le C\sqrt{\varepsilon}$$

such that "$\Omega\cap\partial E\cap C^{\pi(\nu)}_{x,\beta r}=\mathrm{graph}(u,D^{\pi(\nu)}_{x,\beta r})+\nu^\perp$"



Rmk. Here we take advantage again of the invariance of our class of integrands & minimizers to linearize on unif. elliptic eqns with homogeneous Neumann condition

$\mathrm{exc}(E,x,r) < \varepsilon$ $\Rightarrow$ $\exists\, \nu_0$ s.t. $|\nabla\Phi(x,\nu_0)\cdot e_1| < \varepsilon$

$(\Lambda = 0)$

$$\frac{1}{r^{n-1}}\int_{\partial E\cap\Omega\cap B_{x,r}} |\nu_E - \nu_0|^2 < \varepsilon$$

Up to change Φ & E by an affine map fixing Ω we can take $\nu_0 = e_n$

$$\begin{cases} |\nabla\Phi(x,e_n)\cdot e_1| < \varepsilon \\ \dfrac{1}{r^{n-1}}\displaystyle\int_{\partial E\cap\Omega\cap B_{x,r}} |\nu_E - e_n|^2 < \varepsilon \end{cases} \Rightarrow \exists\, \tilde\nu \text{ s.t. } \begin{array}{l} \nabla\Phi(x,\tilde\nu)\cdot e_1 = 0 \\ |e_n - \tilde\nu| < C\varepsilon \end{array}$$

Changing Φ & E once more

$$\begin{cases} \nabla\Phi(x,e_n)\cdot e_1 = 0 \\ \dfrac{1}{r^{n-1}}\displaystyle\int_{\partial E\cap\Omega\cap B_{x,r}} |\nu_E - e_n|^2 < \varepsilon \end{cases}$$

$\Rightarrow \exists\, u : D_{x,r}\cap\Omega \to \mathbb{R}$ Lipschitz

$$\mathcal{H}^{n-1}\big(C_{x,r}\cap\Omega\cap(\partial E\,\Delta\,\mathrm{graph}\, u)\big) < C\varepsilon$$

$$\frac{1}{r^{n-1}}\int_{D_{x,r}\cap\Omega} |\nabla u|^2 < C\varepsilon$$

$$\frac{1}{r^{n-1}}\int_{D_{x,r}\cap\Omega} \nabla^2\Phi(x,e_n)\big[(\nabla u,0),(\nabla\varphi,0)\big] < C\,\mathrm{Lip}(\varphi)\,\varepsilon$$

whenever $\varphi = 0$ on $\partial D_{x,r}\cap\Omega$

$\Rightarrow$ transfer elliptic estimates to $\Omega\cap\partial E$ via u to show that $\mathrm{exc}(E,x,\beta r) \le C\beta^2\,\mathrm{exc}(E,x,r)$

Step four: By step three, charact. of singular set.

With $M = cl(\Omega \cap \partial E)$, we have

$$\Sigma = \{x \in \partial\Omega \cap M : \liminf_{r \to 0} exc(E, x, r) \geq \varepsilon(n, \lambda)\}$$

To prove $\mathcal{H}^{n-2}(\Sigma) = 0$: for $\mathcal{H}^{n-2}$ a.e. $x \in \partial\Omega \cap M$
$\exists\, \nu \in S^{n-1}$, $r_h \searrow 0$ $h \to \infty$ s.t.

$$E^{x, r_h} = \frac{E - x}{r_h} \xrightarrow{L^1} \{y \in \mathbb{R}^n : y \cdot \nu < 0\} \quad (h \to \infty)$$

(i) $\partial E \cap \partial\Omega$ is of finite perimeter in $\partial\Omega = \{x_1 = 0\}$.

(Comparison, needs $\Phi(\cdot, \nu) \in Lip$ & $\Lambda|E \Delta F|$ in min.)

(ii) By De Giorgi rectifiability theorem, for $\mathcal{H}^{n-2}$ a.e. $x \in \partial(\partial E \cap \partial\Omega)$ one has (up to rotations)

$$(\partial\Omega \cap \partial E)^{x,r} \xrightarrow{L^1} e_1^\perp \cap e_n^\perp \quad \text{as } r \to 0$$

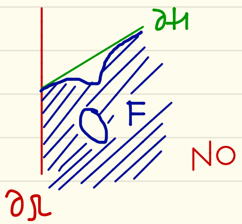
$\Rightarrow$ every blow-up limit of E at x has $e_1^\perp \cap e_n^\perp$ as its trace on $\partial\Omega$

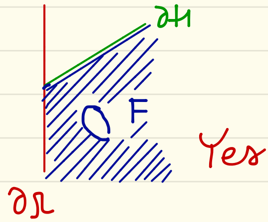
$\Rightarrow$ every blowup lim of E at $x \subseteq \{y \in \mathbb{R}^n : |y \cdot e_n| < L(x \cdot e_1)\}$

$L = L(n, \lambda)$ ← WEDGE

Thus we can freely consider blowups of maximal & of minimal slope (Hardt).

Let F be the blowup of maximal slope. (We don't know if F is a cone.) Let $H = \{y \in \Omega : y \cdot \nu < 0\}$ minimal s.t. $F \subseteq H$, then $\partial H \subseteq \partial F$ by Hopf lemma.





Now $\begin{cases} \partial H \subseteq \partial F \\ F \subseteq H \\ F \text{ min} \end{cases} \Longrightarrow H \text{ super-min.} \Rightarrow \nabla\Phi(\nu)\cdot e_1 \geq 0$

Next minimize the slope among blowup limits of F to find $\tilde{F}$ blow-up limit of F (thus of E). We have $\tilde{F} \subseteq H$. Moreover we find $\tilde{\nu}$ st. $\tilde{H} = \{y \in \Omega : y \cdot \tilde{\nu} < 0\}$ satisfies $\tilde{H} \subseteq \tilde{F}$, and, by Hopf, $\partial\tilde{H} \subseteq \partial\tilde{F} \Rightarrow \tilde{H}$ sub min $\Rightarrow \nabla\Phi(\tilde{\nu})\cdot e_1 \leq 0$.

Thus $\left.\begin{array}{l} \tilde{H} \subseteq H \Rightarrow \tilde{\nu}\cdot e_1 \geq \nu \cdot e_1 \\ \nabla\Phi(\nu)\cdot e_1 \geq 0 \\ \nabla\Phi(\tilde{\nu})\cdot e_1 \leq 0 \end{array}\right\} \Longrightarrow \nu = \tilde{\nu},\ H = \tilde{H} = \tilde{F}.$

Step five: Improve to $\mathcal{H}^{n-3}(\Sigma)=0$. (Almgren's method)

Let $\mathcal{E}=\{\Phi=\Phi(\nu),\ \Phi\ \lambda\text{-elliptic}\}$

$\mathcal{E}^*=\{\Phi\in\mathcal{E}:\ \mathcal{H}^{n-3}(\Sigma_E)=0\ \ \forall E\ \Lambda\text{-min of }\Phi\}$

(i) $\mathcal{E}^*\neq\emptyset$, as $\Phi=|\nu|\in\mathcal{E}^*$. (Taylor or step four + monotonicity)

(ii) $\mathcal{E}^*$ is open in $C^{2,1}$. (By contradiction & blow-up ... standard)

(iii) $\mathcal{E}^*$ is closed in $C^{2,1}$. This is based on the following idea:

$\exists\, C(n,\lambda)$ s.t. given $\Phi\in\mathcal{E}$ and E min. one has

$$\mathcal{H}^{n-3}(\Sigma_E)=0 \iff \int_{\Omega\cap\partial E}|\mathrm{II}_E|^2\leq C(n,\lambda).$$

$\Rightarrow$ $\mathcal{E}=\mathcal{E}^*$.

The proof is thus complete. #

$\Leftarrow$ uses $\mathcal{H}^{n-2}(\Sigma_E)=0$

Fine. Grazie.