A quantitative version of the isoperimetric inequality

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The isoperimetric inequality states that, given a Borel set E of $\mathbb{R}^n$, $n \geq 2$, with finite Lebesgue measure $|E|$, its (distributional) perimeter $P(E)$ is greater or equal than the perimeter of a ball having the same volume as E. That is, if ω_n is the measure of the unit ball B of $\mathbb{R}^n$, we have

$$P(E) \geq n\omega_n^{1/n}|E|^{(n-1)/n}\,, \tag{1}$$

with equality if and only if $E = x + r_E B$ for some $x \in \mathbb{R}^n$ and $r_E := (|E|/\omega_n)^{1/n}$.

In a quantitative version of inequality (1) the isoperimetric deficit $D(E)$,

$$D(E) := \frac{P(E)}{n\omega_n^{1/n}|E|^{(n-1)/n}} - 1\,, \quad |E| > 0\,,$$

controls the distance of E from the set of balls $\{x + r_E B : x \in \mathbb{R}^n\}$. If we restrict our attention to the class of convex sets E it is natural to work with the Hausdorff distance, and the corresponding quantitative inequalities have been studied in depth, among others, by Bernstein [1], Bonnesen [2] (when $n = 2$) and Fuglede [4] (for $n \geq 2$). In the general case, instead, it is natural to adopt the Vitali distance $d(E, F) := |E\Delta F|$, defined as the Lebesgue measure of the symmetric difference between E and F, and introduce the notion of asymmetry of E as

$$A(E) := \inf\left\{\frac{d(E, x + r_E B)}{|E|} : x \in \mathbb{R}^n\right\}.$$

In this setting, a quantitative isoperimetric inequality was shown by Hall, Hayman and Weitsman [8] and Hall [7]. They prove that

$$A(E) \leq C(n)D(E)^{1/4}\,, \quad \text{i.e. } P(E) \geq n\omega_n^{1/n}|E|^{(n-1)/n}\left\{1 + \left(\frac{A(E)}{C(n)}\right)^4\right\}, \tag{2}$$

(here and in the following, $C(n)$ is a constant depending only on the dimension n and possibly changing its value from line to line). A stronger result, in terms of decay rate of A with respect to D, is in fact contained in Hall's paper [7], where it is shown that

$$A(E) \leq C(n)D(E)^{1/2}\,, \quad \text{whenever } E \text{ is axially symmetric.} \tag{3}$$

The decay rate here is sharp, as one can check considering the ellipses $E(r) := \{x \in \mathbb{R}^n : (rx_1)^2 + \sum_{i=2}^n x_i^2 = 1\}$ in the limit $r \to 1$. Hall conjectures the validity of (3) on arbitrary sets, i.e. that

$$A(E) \leq C(n)D(E)^{1/2}\,, \quad \text{for every Borel set } E. \tag{4}$$

In [5] we prove (4), in the way explained below.

Without loss of generality it is assumed that $|E| = \omega_n$. Furthermore, as $A(E) \leq 2$, up to taking $C(n) \geq 2/\sqrt{\delta(n)}$, one can assume that $D(E) \leq \delta(n)$ for some fixed $\delta(n)$. One can prove that $A(E) \to 0$ when $D(E) \to 0$, and this implies that E is somehow close to a ball, in a soft qualitative way, provided we choose $\delta(n)$ small enough. We try to replace E with a "more symmetric" set E', in such a way that the validity of (4) on E can be deduced from the validity of (4) on E'. This amounts in proving that

$$A(E) \leq C(n)A(E')\,, \qquad D(E') \leq C(n)D(E)\,. \tag{5}$$

If the set E' is obtained from E by a symmetrization procedure, one usually gets the second inequality for free (possibly with constant 1, if the given symmetrization decreases the perimeter and leaves the Lebesgue measure unchanged); however, on symmetrizing, we expect to lower the asymmetry too, so that the two inequalities are somehow in competition.

This kind of approach is adopted in [8]. They prove that, given E, a direction ν can be found so that E^*, the Schwarz symmetrization of E with respect to ν, satisfies

$$A(E) \leq C(n)A(E^*)^{1/2}\,. \tag{6}$$

Recall that E^* is the set which intersection E^*_t with $\{x \cdot \nu = t\}$ is a $(n-1)$-dimensional ball centered at $t\nu$ and $\mathcal{H}^{n-1}$-measure equal to $\mathcal{H}^{n-1}(E_t)$, where $E_t := E \cap \{x \cdot \nu = t\}$. The set E^* is axially symmetric and satisfies $P(E^*) \leq P(E)$ (thus $D(E^*) \leq D(E)$). The existence of ν such that (6) holds is clearly a non trivial fact, as one can easily produce a set E such that $A(E) > 0$ but $E^* = B$ with respect to a given ν.

By applying (3) to E^* one finds $A(E^*) \leq C(n)D(E^*)^{1/2} \leq C(n)D(E)^{1/2}$, so deriving (2) from (6). However, being the exponent $1/2$ in (6) optimal, Hall's conjecture cannot be proved this way.

The key notion of our approach is that of n-symmetric set. We say that a set E is n-symmetric if it is invariant by reflection with respect to the n coordinate hyperplanes. The crucial consequence of this definition is that the minimization problem defining $A(E)$ can be somehow trivialized. Indeed, if E is n-symmetric then a simple symmetry argument shows that

$$A(E) = \inf_{x \in \mathbb{R}^n} \frac{d(E, x+B)}{\omega_n} \leq \frac{d(E,B)}{\omega_n} \leq 3A(E)\,. \tag{7}$$

This property allows to prove (4) by induction on the class of n-symmetric sets. Indeed if E is n-symmetric and E^* is its Schwarz symmetrization with respect to, say, the x_1-axis, then

$$\omega_n A(E) \leq d(E,B) \leq d(E,E^*) + d(E^*,B)\,.$$

Since E is n-symmetric, E^* is n-symmetric too. Therefore by applying (7) and (3) to E^* we find

$$d(E^*,B) \leq 3\omega_n A(E^*) \leq C(n)D(E^*)^{1/2} \leq C(n)D(E)^{1/2}\,.$$

On the other hand, $E_t = E \cap \{x_1 = t\}$ is a $(n-1)$-symmetric set in $\{x_1 = t\}$, while E^*_t is an $(n-1)$-dimensional ball centered at the center of symmetry of E_t, and with the same $\mathcal{H}^{n-1}$-measure. Thus, again by (7),

$$d(E,E^*)=\int_{\mathbb{R}}\mathcal{H}^{n-1}(E_t\Delta E_t^*)dt\le 3\int_{\mathbb{R}}\mathcal{H}^{n-1}(E_t)A_{\mathbb{R}^{n-1}}(E_t)dt\,,$$

where $A_{\mathbb{R}^{n-1}}$ denotes the asymmetry in $\{x_1=t\}$. If $D_{\mathbb{R}^{n-1}}$ is the corresponding notion of isoperimetric deficit, by induction one finds

$$\int_{\mathbb{R}}\mathcal{H}^{n-1}(E_t)A_{\mathbb{R}^{n-1}}(E_t)dt\le C(n)\int_{\mathbb{R}}\mathcal{H}^{n-1}(E_t)\sqrt{D_{\mathbb{R}^{n-1}}(E_t)}dt\,. \tag{8}$$

In turn this last quantity is controlled by $D(E)^{1/2}$. This can be heuristically justified by recalling that, if we define $v(t)=\mathcal{H}^{n-1}(E_t)$, $p(t)=\mathcal{H}^{n-2}(\partial E_t)$ and

$$q(t)=\mathcal{H}^{n-2}(\partial E_t^*)=(n-1)\omega_{n-1}^{1/(n-1)}v(t)^{(n-2)/(n-1)}\,,$$

then, by the Coarea Formula,

$$P(E)\ge\int_{\mathbb{R}}\sqrt{v'(t)^2+p(t)^2}dt\,,\qquad P(E^*)=\int_{\mathbb{R}}\sqrt{v'(t)^2+q(t)^2}dt\,.$$

As $P(E^*)\ge P(B)$ by the isoperimetric inequality, we have

$$\begin{aligned}P(B)D(E)&=P(E)-P(B)\ge P(E)-P(E^*)\\&\ge\int_{\mathbb{R}}\sqrt{(v')^2+q^2(1+D_{\mathbb{R}^{n-1}}(E_t))^2}-\sqrt{(v')^2+q^2}\,dt\\&\gtrapprox\int_{\mathbb{R}}q(t)^2D_{\mathbb{R}^{n-1}}(E_t)dt\,,\end{aligned} \tag{9}$$

so that, loosely speaking, one passes from the last term in (9) to the one in (8) by Hölder inequality. To make these arguments completely rigorous a crucial role is played by the aforementioned assumption $D(E)\le\delta(n)$, but this is too technical to be further discussed in here.

Summarizing, n-symmetric sets have some special properties that allow to deduce from (3) that

$$A(E)\le C(n)D(E)^{1/2}\quad\text{if } E \text{ is } n\text{-symmetric.} \tag{10}$$

In turn we can deduce (4) from (10) once we show that, given a set E, then a n-symmetric set E' can be found so that (5) holds true. We now pass to discuss this last step. We start by considering a simpler task, i.e. we just ask E' to be symmetric with respect to one hyperplane, say $\{x_1=0\}$. Up to translating E in the x_1-direction we achieve $|E\cap\{x_1>0\}|=|E\cap\{x_1<0\}|$. If we denote by E_1^+ the set obtained by reflecting $E\cap\{x_1>0\}$ w.r.t. $\{x_1=0\}$, and similarly define E_1^-, then $E_1^\pm$ are both symmetric with respect to $\{x_1=0\}$, have the same measure as E and satisfy $P(E_1^+)+P(E_1^-)\le 2P(E)$. Therefore $D(E_1^\pm)\le 2D(E)$, and the second inequality in (5) is certainly achieved. On the other hand it could be as well that $A(E)>0$ but $A(E_1^\pm)=0$, if for example

$$E=\big[B\cap\{x_1>0\}\big]\cup\big[(B+e_2)\cap\{x_1<0\}\big]\,. \tag{11}$$

Note that this set E exhibit the bad behavior with respect to symmetrization by reflection *only* in the x_1-direction. Luckily enough, this is a general fact, and one can prove that given two coordinate directions, say x_1 and x_2, and considered the four sets $E_1^{\pm}, E_2^{\pm}$, then there exists at least one set E' among them such that $A(E) \leq C(n)A(E')$. Being certainly $D(E') \leq 2D(E)$, we have found E' satisfying (5), and having an hyperplane of symmetry.

This procedure can be applied $(n-1)$-times so to find (up to a possible final rotation) a set E' symmetric with respect to the first $(n-1)$ coordinate hyperplanes and such that (5) holds. At this stage we are forced to symmetrize E' with respect to the x_n-direction, and clearly the above selection argument cannot be repeated further without possibly stepping into a loop. However, it comes out that one among $(E')_n^+$ and $(E')_n^-$ (defined in the obvious way after translating E' so that $|E' \cap \{x_n > 0\}| = |E' \cap \{x_n > 0\}|$) shall satisfy (5). This is basically due to the fact that being E' already symmetric with respect to $x_1, \dots, x_{n-1}$, it is then impossible to meet in the x_n-direction the situation exemplified by (11).

Apart from being useful in proving inequality (4), these kind of arguments, and especially the notion of n-symmetry, can be effectively used in the study of quantitative versions of the Sobolev inequalities

$$S(n,p)\left(\int_{\mathbb{R}^n} |f|^{np/(n-p)}\right)^{(n-p)/np} \leq \left(\int_{\mathbb{R}^n} |\nabla f|^p\right)^{1/p},$$

for $1 \leq p < n$. The cases $p = 1$ and $1 < p < n$ are of course quite different, and are considered, respectively, in [6] and [3].

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